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## **FINITE ELEMENT ANALYSIS OF COMPRESSOR BLADES UNDER EXTENSION, BENDING, AND TORSION LOADS**

### **Part II: Anisotropic Materials**

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#### **ABSTRACT**

The objective of this research is to develop a finite element model for the analysis of the dynamic as well as the static response of a compressor blade made of composite materials subjected to extension, transverse, and torsion loads. The equation of motion is derived based on the variation technique with the principle of the total potential energy of a laminated fiber reinforced structures with different fiber orientation angles. The formulation is based on the classical laminate theory and the warping effect is taken into consideration. A one dimensional linear isoperimetric element with Lagrange and hermit cubic shape function are used to model the axial and transverse deformation. A two end nodes and one intermediate node as well is implemented for modeling the torsion deformation. The bending – torsion and axial coupling are introduced in the stiffness and mass matrices. The obtained results of the present model are compared to the available finite element and analytical results of others investigators, a good agreement is generally obtained.

#### **KEYWORDS**

Finite element analysis- compressor blade design- composite materials mechanics- warping - structural analysis.

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## NOMENCALUTRE

$\varepsilon$	Axial strain.
$\gamma$	shear strain.
$\varphi(x)$	The twist angle at each section.
$\theta$	The bending angle (slope).
$\rho$	Density.
$\omega$	Natural frequency of the vibration.
$\nu$	Poisson's ratio.
$\lambda$	The Warping function.
$\varphi_1, \varphi_2, \text{ and } \varphi_3.$	Torsion nodal displacements.
$\Delta(\lambda)$	polynomial of degree n in $\lambda$ .
$\Omega(x)$	Constant distributed load acting on beam element.
a	Height of cross section of the beam.
b	Width of cross section of the beam.
c1, c2, c3, c4.	Constants of integration.
f <sub>axial</sub> (x)	Axial shape functions.
f <sub>bending</sub> (x)	Bending shape functions.
f <sub>torsion</sub> (x)	Torsion shape functions.
f <sub>Axial-Warping</sub>	The effect of warping on torsion shape functions.
h	Single Element length.
m(x)	Mass per unit length.
m <sub>axial</sub>	Axial mass matrix.
m <sub>bending</sub>	Bending mass matrix.
m <sub>torsion</sub>	Torsion mass matrix.
n	Number of elements.
u1, and u2.	Axial nodal displacements.
w1, $\theta_1$ , w2, $\theta_2$	Bending nodal displacements.
V	Volume.
A	Cross sectional area ( $A = a.b$ ).
A, B, and D	Extension, coupling, and bending matrices.
E	Modulus of Elasticity.
F	Total mechanical Loads on single beam element.
F <sub>xwarping</sub>	The effect of torsional warping on torsional stiffness.
G	Shear modulus.
I <sub>m</sub> (x)	Mass polar moment of inertia per unit length.
I	Second area moment of inertia.
J.	Area Polar moment of inertia.
[K]	Stiffness matrix.
K <sub>axial</sub> .	Axial stiffness matrix.
K <sub>bending</sub> .	Bending stiffness matrix.
K <sub>torsion</sub> .	Torsion stiffness matrix.
K <sub>warping</sub> .	Cross-sectional warping coefficient.
L	Length of the Cantilever beam.
P(x).	Constant axial force acting on beam element.
T(x).	Constant Torque moment acting on beam element.
U <sub>i</sub> .	Internal strain energy of the total system.
We.	External work done on the system.

## INTRODUCTION

The critical structural element of a typical gas turbine engine is the compressor or turbine disk carrying several blades around its circumference. These units operate in severe environments characterized by high speeds of rotation and temperature. Most of the failures reported have been due to vibration-induced fatigue of bladed disk units. The failure of even a single blade in an engine can adversely affect the performance. The impact of composite materials use on jet engine performance is very substantial. The higher order of the thrust-to-weight ratio is achieved with the advanced graphite fiber composite. A classical structural analysis is performed to improve the static and dynamic performance of these critical unites under extension, transverse, and torsion loads.

Several researches have studied the analysis of laminated composite beams. Kapania R. K. et. al [1] proposed an exact solutions for a free vibration analysis of simply supported composite beams based on the classical theory. Vinson J. et. al [2] studied a structures made of composite materias at which the effect of the transverse shear deformation and rotary inertia are neglected. A finite element analysis of symmetrically-laminated beam based on the first-order shear deformation theory was reported by Chen and Yang [3]. The exact solution for a beam with various boundary conditions is presented by Chandrasekhar et. al [4]. It is noted that the references [2, 4] neglect the Poisson effect.

The calculation of the uncoupled modes of arbitrarily shaped cantilever beams under bending load was investigated [5-8], but little work has been done on calculating the coupled modes of such beams. Modelling of beam structures with coupled behaviour is discussed in [9–14]. Sakawa et.al [9] used a shear-in deformable theory to model a mass coupled beam. The internal beam damping was included in their model. The beam was mounted on a rotating shaft, and an actuation torque was provided by a motor. Banks and Smith [10] studied a coupling problem similar to that of Sakawa et.al. However, in their model the warping effects and the internal shear damping were considered. Shen [11] also employed a shear-in deformable theory in which the warping effect was not included in composite beams. Banerjee and Williams [12] studied the vibration of a beam with geometrical coupling. Although they used the shear-deformable theory, warping effect was ignored.

A shear deformable theory was simplified by Sankar [13] to be applicable for one-dimensional beam analysis. St. Venant torsion and warping effects are explicitly included in his model. Sankar's theory shows some identical results to those obtained by Borese et al [14]. Quite good agreement for a laminate beam is obtained between Sankar results and those reported by Tsai et al [15]. It has been shown by Suresh et al [16] that warping effects can significantly influence the natural frequencies of a composite beam, and hence, warping should not be neglected. Whitney [17] has shown that the shear-indeformable theory is not capable of studying the behavior of a laminate in pure torsion. He has also indicated that the effect of shear-deformation on the torsional stiffness for a variety of laminates with ( $b/h < 15$ ) can be important.

In the present work a finite element formulation is developed to study the static and dynamic response for a composite compressor blade made from a fibre reinforced composite material with different stacking sequence subjected to a multi acting loads.

The warping effect is taken into consideration during the study. A MATLAB code is developed to perform the analysis for different examples.

### THEORITICAL FORMULATION

The compressor blade is model as an advanced beam under the assumptions of Euler-Bernoulli theory of beam as shown in Fig.1 and Fig 2. The structure is subjected to axial, transverse, and torsion loads [18]. The torsion deformation of the beam cross section subjected to a torque moment is shown in Fig.3 and Fig 4, and the following formulations are obtained [19].

$$\Delta Y = r \cdot \cos(\beta + \varphi) - r \cdot \cos(\beta) \tag{1}$$

where  $\varphi$  is the twist angle for each section "x". For small angle  $\Phi$ ,  $\cos(\Phi) \cong 1$ ,  $\sin(\Phi) \cong \Phi$ , therefore,

$$\Delta Y = -\varphi \cdot r \cdot \sin(\beta) \tag{2}$$

Since  $\Delta Y = -Z \cdot \varphi = -Z \cdot X \cdot \theta$

$$Z = r \cdot \sin(\beta) \tag{3}$$

Where,  $\theta$  twist is the twist angle per unit length. Similarly,

$$\Delta Z = r \cdot \sin(\beta + \varphi) - r \cdot \sin(\beta) \tag{4}$$

$$\Delta Z = \varphi \cdot r \cdot \cos(\beta) \tag{5}$$

$$\Delta Z = Y \cdot \varphi = Y \cdot X \cdot \theta \tag{6}$$

Thus the assumed displacements field equations based on the classical beam theory at any point in the x, y, and z directions are:

$$\begin{aligned} U(x) &= U_s(x) - z \cdot \frac{\partial w(x)}{\partial x} - y \cdot \frac{\partial \phi(x)}{\partial x} \dots\dots(a) \\ V(x) &= -z \cdot \phi(x) \dots\dots(b) \\ W(x) &= w(x) + y \cdot \phi(x) \dots\dots(c) \end{aligned} \tag{7}$$

where,  $U_s(x)$ ,  $w(x)$ , and  $\phi(x)$  are the axial displacement, the bending displacement, and the torsion twist angle, respectively. The warping deformation of a bar with non circular cross section is shown in Fig.4. The warping function must satisfy following two conditions:

$$\nabla^2 \lambda = \frac{\partial^2 \lambda}{\partial y^2} + \frac{\partial^2 \lambda}{\partial z^2} = 0 \tag{8}$$

$$\left( \frac{\partial \lambda}{\partial y} - z \right) \cdot \frac{dz}{ds} - \left( \frac{\partial \lambda}{\partial z} - y \right) \cdot \frac{dy}{ds} = 0 \tag{9}$$

From these conditions, the warping function can be written as;

$$\lambda = K_{warping} \cdot y \cdot z \quad (10)$$

where,  $K_{warping}$  is the cross-sectional warping coefficient. For rectangular cross section beam  $K_{warping} = 1$

### STRAIN-DISPLACEMENT RELATIONS

The strain displacement relations can be given as;

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial U_s(x)}{\partial x} - z \cdot \frac{\partial^2 w(x)}{\partial x^2} - z \cdot y \cdot \frac{\partial^2 \phi(x)}{\partial x^2} & \gamma_{xy} &= -2z \cdot \frac{\partial \phi(x)}{\partial x} \\ \epsilon_{yy} &= 0 & \epsilon_{zz} &= 0 & \gamma_{zx} &= 0 & \gamma_{zy} &= 0 \end{aligned} \quad (11)$$

### THE LAMINA STRESS STRAIN RELATIONS

The lamina stress-strain relations in the principle material axes is given as [20],

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \cdot \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} \quad (12)$$

where, the reduced stiffness constants in the principal material directions are:

$$\begin{aligned} Q_{11} &= \frac{S_{22}}{S_{11} \cdot S_{22} - S_{12}^2} & Q_{22} &= \frac{S_{11}}{S_{11} \cdot S_{22} - S_{12}^2} & Q_{12} &= \frac{-S_{12}}{S_{11} \cdot S_{22} - S_{12}^2} \\ Q_{21} &= \frac{-S_{21}}{S_{11} \cdot S_{22} - S_{12}^2} & Q_{66} &= G_{12} \end{aligned} \quad (13)$$

And the elastic compliances constants of a lamina are given as;

$$S_{11} = \frac{1}{E_1} \quad S_{22} = \frac{1}{E_2} \quad S_{12} = \frac{-\nu_{12}}{E_1} \quad S_{21} = \frac{-\nu_{21}}{E_2} \quad S_{66} = \frac{1}{G_{12}} \quad (14)$$

The transformed stress-strain relation of a lamina in the structure geometric axes  $x, y$  is given by;

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \quad (15)$$

Where,  $\overline{Q}_{ij}$  is transformed stiffness coefficient [20]:

### VARIATIONAL FORMULATION

The equations of motion can be detected by means of variational approach by equating the internal strain energy and the Virtual work expressions such as [8];

$$\delta U_i = \delta W_e \tag{16}$$

$$\begin{aligned} & \int_V (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{zz} \delta \epsilon_{zz} + \tau_{zx} \delta \gamma_{zx} + \tau_{zy} \delta \gamma_{zy} + \tau_{xy} \delta \gamma_{xy}) \cdot dV \\ & = \int_0^h P(x) \cdot \delta U_s(x) \cdot dx + \int_0^h \Omega(x) \cdot \delta w(x) \cdot dx + \int_0^h T(x) \cdot \delta \phi(x) \cdot dx \end{aligned} \tag{17}$$

The element mass matrix is obtained using the kinetic energy term as;

$$\begin{aligned} T(t) &= \frac{1}{2} \cdot \int_0^L m(x) \cdot v^2 \cdot dx \\ &= \frac{1}{2} \int_0^L \left( m(x) \cdot \dot{U}^2 + m(x) \cdot \dot{w}^2 + \rho \cdot I_x(x) \cdot \dot{\phi}^2 \right) dx \end{aligned} \tag{18}$$

where  $\rho$  is the density of the material,  $v$  is the velocity (displacement differentiation), and  $m(x)$  is the mass per unit length of the beam.

By substituting equations (15) into equation (17) the strain energy expression can be written as;

$$U_d = \iiint \left( \overline{Q}_{11} \cdot \epsilon_{xx}^2 + 2\overline{Q}_{12} \cdot \epsilon_{xx} \cdot \epsilon_{yy} + 2\overline{Q}_{16} \cdot \epsilon_{xx} \cdot \gamma_{xy} + \overline{Q}_{66} \cdot \gamma_{xy}^2 \right) \cdot dx \cdot dy \cdot dz \tag{19}$$

$$U_d = \iiint \left( \overline{Q}_{11} \cdot \left( \frac{\partial U_s(x)}{\partial x} \right)^2 - \overline{Q}_{11} \cdot z^2 \cdot \left( \frac{\partial^2 w(x)}{\partial x^2} \right)^2 - \overline{Q}_{11} \cdot z^2 \cdot y^2 \cdot \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right)^2 \right. \\ \left. - 2z \cdot \overline{Q}_{11} \cdot \left( \frac{\partial U_s(x)}{\partial x} \right) \cdot \left( \frac{\partial^2 w(x)}{\partial x^2} \right) - 2z \cdot y \cdot \overline{Q}_{11} \cdot \left( \frac{\partial U_s(x)}{\partial x} \right) \cdot \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right) \right. \\ \left. + 2z^2 \cdot y \cdot \overline{Q}_{11} \cdot \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right) \cdot \left( \frac{\partial^2 w(x)}{\partial x^2} \right) - 4z \cdot \overline{Q}_{16} \cdot \frac{\partial U_s(x)}{\partial x} \cdot \frac{\partial \phi(x)}{\partial x} \right. \\ \left. + 4z^2 \cdot \overline{Q}_{16} \cdot \frac{\partial^2 w(x)}{\partial x^2} \cdot \frac{\partial \phi(x)}{\partial x} - 4z^2 \cdot y \cdot \overline{Q}_{16} \cdot \frac{\partial^2 \phi(x)}{\partial x^2} \cdot \frac{\partial \phi(x)}{\partial x} \right. \\ \left. + 4\overline{Q}_{66} \cdot z^2 \cdot \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \right) \cdot dx \cdot dy \cdot dz \tag{20}$$

$$U_d = \iint \left( \begin{aligned} & \mathbf{A}_{11} \cdot \left( \frac{\partial U_s(x)}{\partial x} \right)^2 - \mathbf{D}_{11} \cdot \left( \frac{\partial^2 w(x)}{\partial x^2} \right)^2 - \mathbf{D}_{11} \cdot y^2 \cdot \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right)^2 \\ & - 2\mathbf{B}_{11} \cdot \left( \frac{\partial U_s(x)}{\partial x} \right) \cdot \left( \frac{\partial^2 w(x)}{\partial x^2} \right) - 2\mathbf{B}_{11} \cdot y \cdot \left( \frac{\partial U_s(x)}{\partial x} \right) \cdot \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right) \\ & + 2\mathbf{D}_{11} \cdot y \cdot \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right) \cdot \left( \frac{\partial^2 w(x)}{\partial x^2} \right) - 4\mathbf{B}_{16} \cdot \frac{\partial U_s(x)}{\partial x} \cdot \frac{\partial \phi(x)}{\partial x} \\ & + 4\mathbf{D}_{16} \cdot \frac{\partial^2 w(x)}{\partial x^2} \cdot \frac{\partial \phi(x)}{\partial x} - 4\mathbf{D}_{16} \cdot y \cdot \frac{\partial^2 \phi(x)}{\partial x^2} \cdot \frac{\partial \phi(x)}{\partial x} \\ & + 4\mathbf{D}_{66} \cdot \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \end{aligned} \right) \cdot dx \cdot dy \quad (21)$$

The laminates extension, coupling, and bending matrices are given as;

$$\begin{aligned} \mathbf{A}_{ij} &= \sum_{k=1}^N (\overline{\mathbf{Q}}_{ij})_k \cdot (Z_k - Z_{k-1}) \\ \mathbf{B}_{ij} &= \frac{1}{2} \sum_{k=1}^N (\overline{\mathbf{Q}}_{ij})_k \cdot (Z_k^2 - Z_{k-1}^2) \\ \mathbf{D}_{ij} &= \frac{1}{3} \sum_{k=1}^N (\overline{\mathbf{Q}}_{ij})_k \cdot (Z_k^3 - Z_{k-1}^3) \end{aligned} \quad (22)$$

### FINITE ELEMENT MODELING

A Lagrange shape functions for the axial deformation  $U_s(x)$  is used [21];

$$\mathbf{f}_{axial} = \left[ \left[ 1 - \frac{x}{h} \right] \quad \left[ \frac{x}{h} \right] \right] \quad (23)$$

The transverse deformation  $w(x)$  is expressed in terms of a hermit cubic shape function [21]

$$\mathbf{f}_{bending} = \left[ \left[ 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3 \right] \quad \left[ \frac{x}{h} - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3 \right] \quad \left[ 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3 \right] \quad \left[ -\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3 \right] \right] \quad (24)$$

For the three nodal displacements on beam element as shown in Fig 5., the shape function of the Torsion displacements  $\phi(x)$  can be expressed as [19,21],

$$\mathbf{f}_{torsion} = \left[ \left[ 2\left(\frac{x}{h}\right)^2 - 3\left(\frac{x}{h}\right) + 1 \right] \quad \left[ -4\left(\frac{x}{h}\right)^2 + 4\left(\frac{x}{h}\right) \right] \quad \left[ 2\left(\frac{x}{h}\right)^2 - \left(\frac{x}{h}\right) \right] \right] \quad (25)$$

The Nodal displacements for axial, bending and torsion displacements are termed as; "u1, u3", "w1, θ1, w3, θ3" and "φ1, φ2, φ3" respectively, which are illustrated in the Fig.5 and Fig.6. By inserting the shape functions into the displacement equations (7), the displacement equation can be written as,

$$U(x) = f_x \cdot [q_i] = (1-s)u_1 - \frac{z}{h}(-6s + 6s^2)w_1 - z(1-4s + 3s^2)\theta_1 - \frac{y \cdot z}{h}(4s-3)\phi_1 + (s)u_3 - \frac{y \cdot z}{h}(-8s + 4)\phi_2 - \frac{z}{h}(6s - 6s^2)w_3 - z(-2s + 3s^2)\theta_3 - \frac{y \cdot z}{h}(4s-1)\phi_3 \dots\dots\dots(a)$$

$$V(x) = f_y \cdot [q_i] = -z(2s^2 - 3s + 1)\phi_1 - z(-4s^2 + 4s)\phi_2 - z(2s^2 - s)\phi_3 \dots\dots\dots(b)$$

$$W(x) = f_z \cdot [q_i] = (1-3s^2 + 2s^3)w_1 + h(s-2s^2 + s^3)\theta_1 + y(2s^2 - 3s + 1)\phi_1 + y(-4s^2 + 4s)\phi_2 + (3s^2 - 2s^3)w_3 + h(-s^2 + s^3)\theta_3 + y(2s^2 - s)\phi_3 \dots\dots\dots(c)$$

(26)

where fx, fy and fz are the combined shape function of the axial, bending and torsion displacement equations, and the parameter s = x/h

**EQUATION OF MOTIOIN**

By differentiating the displacement field equations (26), and substituting into equations (17), (18), and (21) the element force vector one can obtain;

$$[M]_e \cdot [\ddot{q}] + [K]_e \cdot [q] = [F]_e \tag{27}$$

where  $[M]_e$ ,  $[K]_e$ , and  $[F]_e$  are the element mass matrix, the element stiffness matrix and the element force vector given in the Appendix.

Assembling the mass, stiffness matrices and the load vector for the whole structure. The global equation of motion is expressed as;

$$[M] \cdot [\ddot{q}] + [K] \cdot [q] = [F] \tag{28}$$

For free vibration analysis the right hand side of equation (28) is equal to zero [15].

A MATLAB code is developed to check the present model for a certain structure with different boundary condition and different geometry and materials properties.

**VALIDATION EXAMPLE**

A static and dynamic response of a graphite epoxy composite beam subjected to a transverse load with the following properties is proposed to validate the model. The dimensionless material properties of a laminated composite beam;

$$E1/E2 = 25, G12 = 0.5(E2), \text{ and } \nu12 = 0.25.$$



The composite beam dimensionless geometry (length/thickness), are  $L/a = 5, 10, 50$ . It is subjected to uniformly distributed transverse load = 10000 N/m. The beam is solved for a case of symmetric cross-ply ( $0^\circ/90^\circ/0^\circ$ ) and the case of anti-symmetric cross-ply ( $0^\circ/90^\circ$ ). All laminas are assumed to be made of the same material and with the same thickness. The bending deflections are calculated in non-dimensional form as follows:

$$\frac{w}{W} = \frac{W \cdot A \cdot E_2 \cdot a^2 \cdot 10^2}{\Omega \cdot L^4} \tag{29}$$

where  $W$  is the tip transverse deflection of the beam,  $A$  is the beam cross-section area,  $\Omega$  is the uniformly distributed transverse load.

Table (1): Non-dimensional central tip deflection for symmetric cross-ply laminated composite cantilever beam ( $0^\circ/90^\circ/0^\circ$ )

L/a	Non-dimensional central tip deflection				
	HOBT	SOBT	FOBT	CBT	Present Model
5	6.824	5.948	6.698	2.198	2.19783
10	3.455	3.315	3.323	2.198	2.19783
50	2.251	2.235	2.243	2.198	2.19783

Table (2): Non-dimensional central tip deflection for anti-symmetric cross-ply laminated composite cantilever beam ( $0^\circ/90^\circ$ )

L/a	Non-dimensional central tip deflection				
	HOBT	SOBT	FOBT	CBT	Present Model
5	15.279	15.695	16.436	11.293	11.28776
10	12.343	12.400	12.579	11.293	11.28776
50	11.337	11.338	11.345	11.293	11.28776

Tables (1) and (2) show the model non-dimensional deflections to be very close to the values reported in reference [22], where HOBT, SOBT, FOBT, and CBT stand for high order beam theory, second order beam theory, first order beam theory, and classical beam theory respectively.

To test the dynamic response for graphite-epoxy composite cantilever beam. The material properties are as follows [23];

$$E_1 = 144.8 \text{ GPa.}, \quad E_2 = 9.65 \text{ GPa.}, \quad G_{12} = 4.14 \text{ GPa.},$$

$$\nu_{12} = 0.3, \quad \rho_{\text{comp}} = 1389.23 \text{ kg/m}^3.$$

The composite beam with a geometry such as  $L/a = 15$  and symmetric cross-ply lamina with stacking sequence ( $0^\circ/90^\circ/90^\circ/0^\circ$ ) and ( $45^\circ/-45^\circ/-45^\circ/45^\circ$ ). The fundamental natural frequencies are given in non-dimensional form as follows:

$$\bar{\omega} = \omega \cdot L^2 \sqrt{\frac{\rho_{comp}}{E_1 \cdot a^2}} \tag{30}$$

where  $\omega$  is the free vibration natural frequency,  $\rho_{comp}$  is the composite material density.

Table (3): Non-dimensional natural frequencies of different symmetric cross-ply laminated composite cantilever beam.

Composite beam layer orientation	FSDT	HSDT	Present Model
(0°/90°/90°/0°)	0.9231	0.9241	0.95527
(45°/-45°/-45°/45°)	0.5551	0.5965	0.56187

Table (3) shows the model non-dimensional natural frequencies compared by results introduced reference [23].

The obtained results in both static and dynamic cases are accepted such that the difference in static deflection of classical beam theory (CBT) of symmetric cross-ply (0°/90°/0°) cantilever beam for different L/a ratios is found to be 0.0076 % while for anti-symmetric cross-ply (0°/90°) cantilever beam, the difference between the static deflection calculated and the CBT is found to be 0.046 %. In dynamic free vibration, the non-dimensional natural frequency for symmetric cross-ply (0°/90°/90°/0°) compared with FSBT and the difference was 3.48 %, and for symmetric cross-ply (45°/-45°/-45°/45°) compared with FSBT, the difference was 1.219 %. In spite of that the present model is taking the warping effect into consideration.

To validate the static and dynamic responses for the model under a multi-acting loads (axial – transverse – torsion), a graphite/epoxy T300/934 composite beam [45 /-45 /-45 /45] with the following properties are examined.

- $\nu = 0.3$ ;  $\rho = 1389.23 \text{ kg/m}^3$ ;  $G_{12} = 4.14 \times 10^9 \text{ N/m}^2$ ,
- $E_1 = 144.8 \times 10^9 \text{ N/m}^2$ ;  $E_2 = 9.65 \times 10^6 \text{ N/m}^2$ ;
- $L = 0.1524 \text{ m}$  ;  $b = 0.06096 \text{ m}$  ;  $a = 0.03048 \text{ m}$
- $P(x) = 100 \text{ N/m}$  ;  $\Omega(x) = -10000 \text{ N/m}$  ;  $T = 1000 \text{ N.m/m}$

The coupled natural frequencies and tip deflection for a composite beam for a different beam lengths are shown in Table (4).

Table (4): The natural frequencies of T300/934 symmetric cross-ply laminated composite cantilever beam [45 /-45 /-45 /45]

L in mm	b/a	Tip deflection in mm	modes	$\omega_{coupling}$ (rad/sec)	$f_{coupling}$ (Hz)
100	1	-0.003375	1	56330.8	8965.32527
			2	355118.29	56518.8312
			3	975434.6	155245.238

L in mm	b/a	Tip deflection in mm	modes	$\omega_{\text{coupling}}$ (rad/sec)	$f_{\text{coupling}}$ (Hz)	
200	2	-0.001687	1	57565.37	9161.81319	
			2	354715.16	56454.6711	
			3	977232.07	155531.314	
	3	-0.00112506	1	56810	9041.59232	
			2	356089.05	56673.3325	
			3	975953.82	155327.875	
	300	1	-0.0033752	1	28784.4	4581.17954
				2	177371.92	28229.6178
				3	488331.24	77720.3307
2		-0.0016876	1	28706.57	4568.79251	
			2	177437.91	28240.1205	
			3	488390.2	77729.7145	
3		-0.001125	1	28662.2	4561.73081	
			2	177462.28	28243.9991	
			3	488353.67	77723.9006	
300	1	-0.0033752	1	19114.034	3042.09299	
			2	118327.05	18832.3349	
			3	325486.82	51802.8363	
	2	-0.0016876	1	19109.76	3041.41277	
			2	118311.47	18829.8553	
			3	325563.34	51815.0149	
	3	-0.001125	1	19100.0	3039.85941	
			2	118343.93	18835.0214	
			3	325531.16	51809.8932	

Fig.7 shows the axial, transverse and torsion deformation on 3-D drawing. The bending mode shapes from 1st to 5th are shown in Fig.8 up to Fig.12. The bending torsion coupling mode shapes is illustrated from Fig.13 up to Fig.17. The coupled natural frequencies are shown for different beam lengths

**CONCLUSION**

A finite element model for solving a compressor blade made of fiber reinforced composite materials was investigated. The structure was subjected to a multi acting loads. The warping deformation was taken into consideration. The torsion-bending and axial coupling was shown in mass and stiffness matrices. The obtained results were found much closed to other theories FOSD and HOSD etc. The coupled torsion, bending, and axial deformation for static deflection as well as the modes of the free vibration analysis was computed. A higher order shear deformation theories for the assumed displacement field equations is proposed to be a future work.

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**FIGURES**

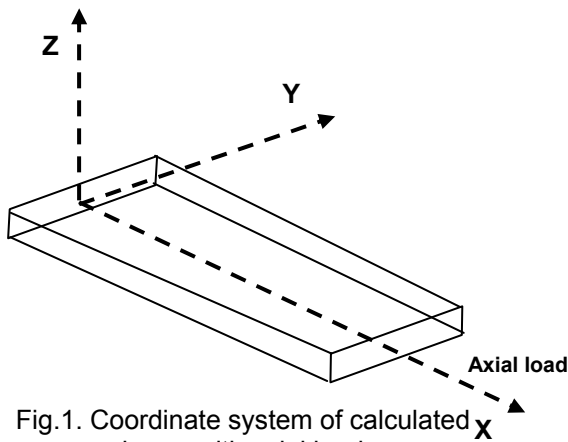


Fig.1. Coordinate system of calculated beam with axial load

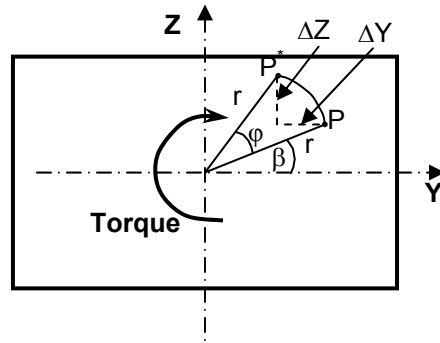


Fig.3. Torsion displacements

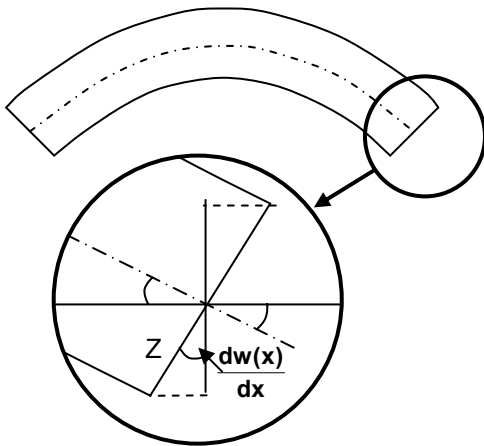


Fig.2. Bending displacement in axial direction

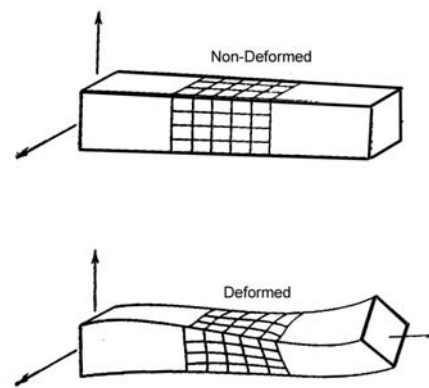


Fig.4. Warping of bar of non-circular cross section Under torsion loading

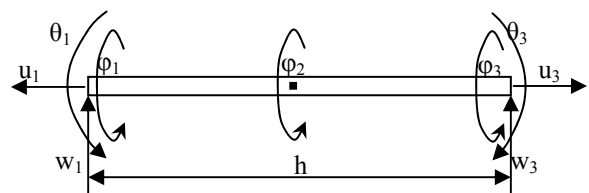


Fig.5. Nodal Displacements of axial-bending-torsion Element

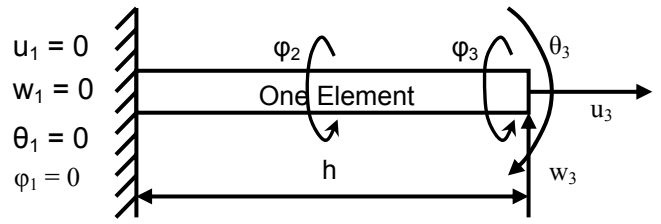


Fig.6. Degrees freedoms of Fixed-Free beam

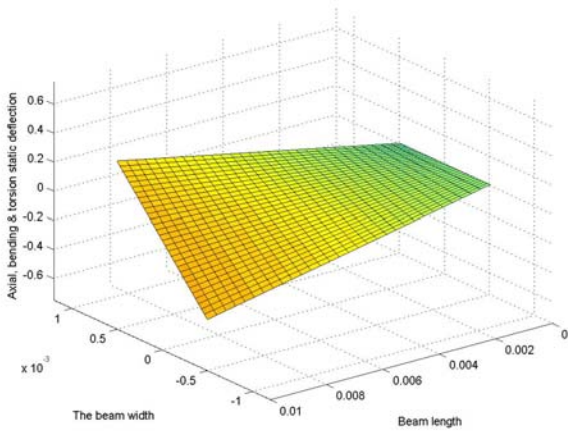


Fig.7. The static coupled bending-torsion deflection

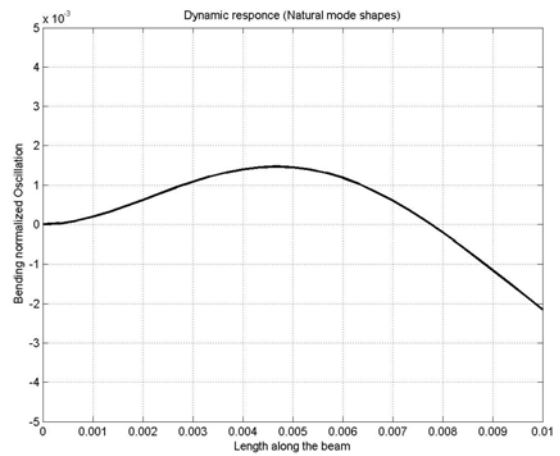


Fig.9. The Second bending mode shape

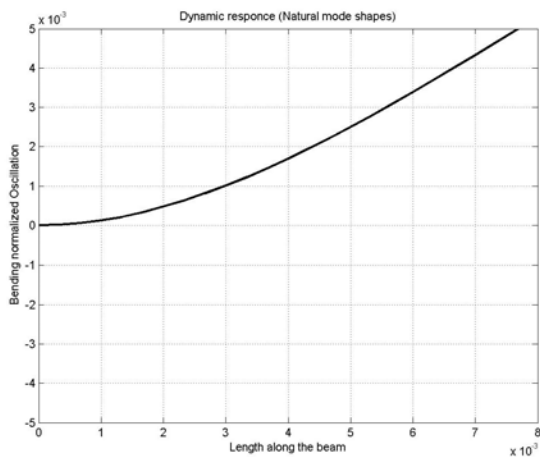


Fig.8. The First bending mode shape

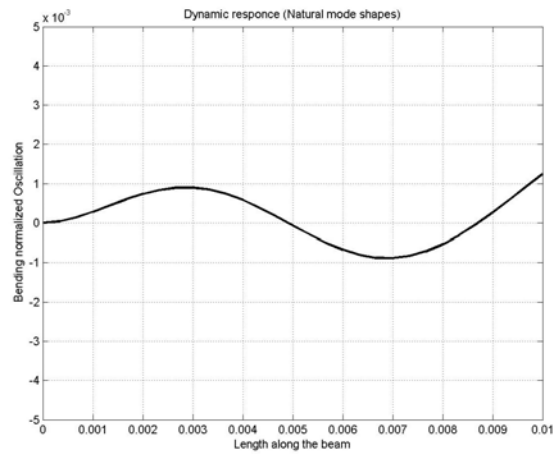


Fig.10. The Third bending mode shape

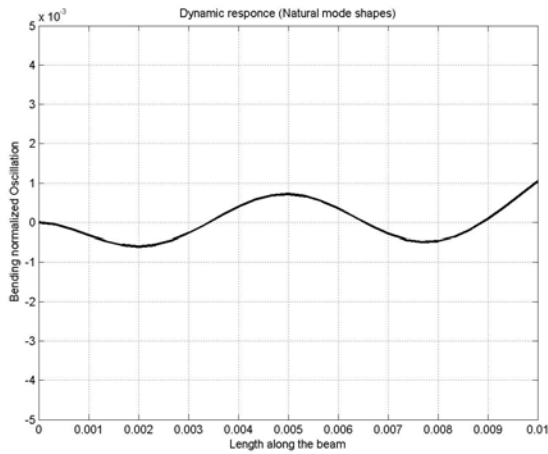


Fig.11. The Fourth bending mode shape

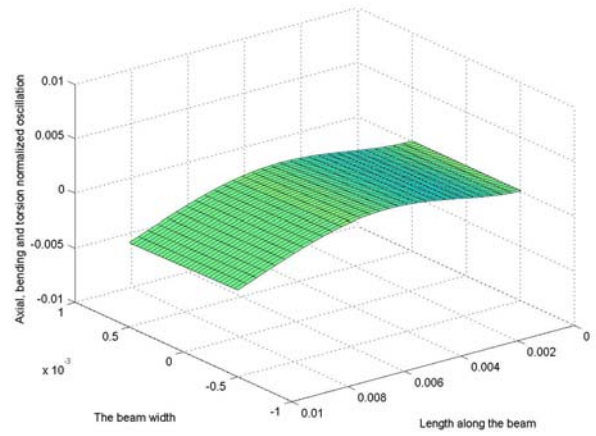


Fig.14. The second bending-torsion coupling mode shape

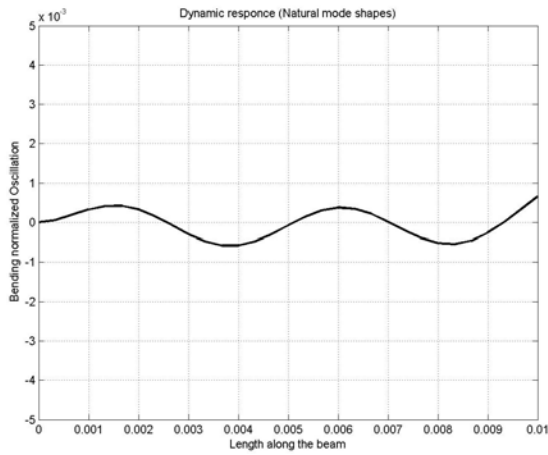


Fig.12. The Fifth bending mode shape

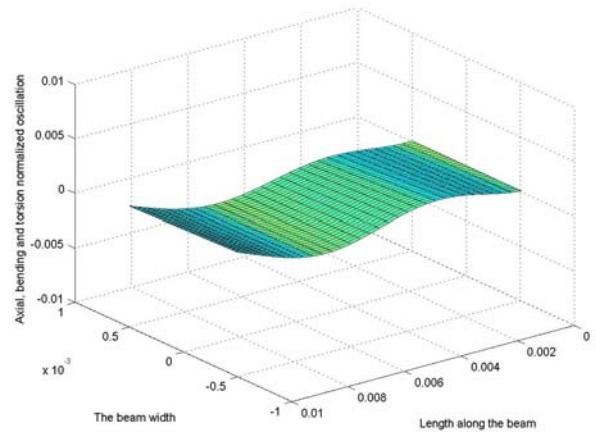


Fig.15. The third bending-torsion coupling mode shape

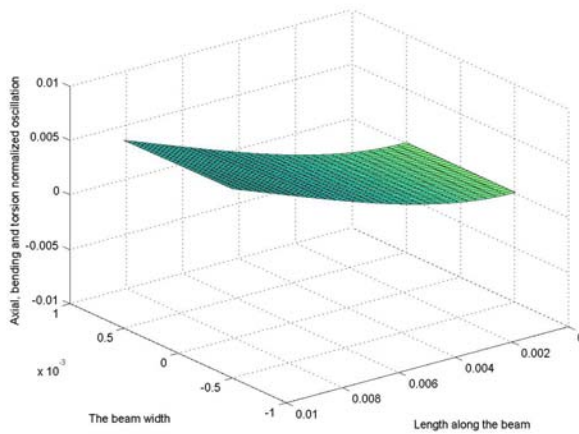


Fig.13. The first bending-torsion coupling mode shape

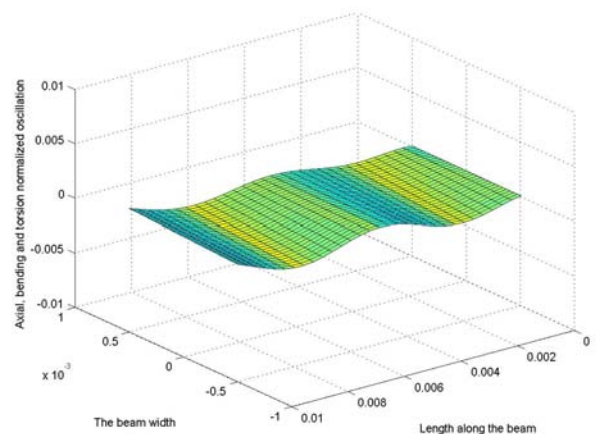
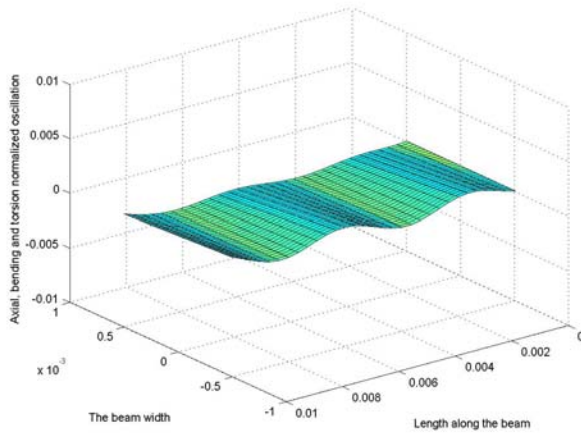


Fig.16. The fourth bending-torsion coupling mode shape

Fig.17. The fifth bending-torsion coupling mode shape



Appendix:

The element stiffness matrix  $[K]_e$  :

$$\begin{aligned}
 K_{11} &= \int_A \frac{A_{11}}{L^2} \cdot dA & K_{12} &= \int_A \frac{D_{11}}{L} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \cdot dA & K_{13} &= \int_A B_{11} \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \cdot dA \\
 K_{14} &= \int_A \frac{4B_{11}y}{L^3} + \frac{2B_{16}}{L} \left( \frac{4x}{L^2} - \frac{3}{L} \right) \cdot dA & K_{15} &= \int_A \frac{-8B_{11}y}{L^3} + \frac{2B_{16}}{L} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \cdot dA \\
 K_{16} &= \int_A \frac{-A_{11}}{L^2} \cdot dA & K_{17} &= \int_A \frac{B_{11}}{L} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \cdot dA & K_{18} &= \int_A B_{11} \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \cdot dA \\
 K_{19} &= \int_A \frac{4B_{11}y}{L^3} + \frac{2B_{16}}{L} \left( \frac{4x}{L^2} - \frac{3}{L} \right) \cdot dA & K_{22} &= \int_A D_{11} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \cdot dA \\
 K_{23} &= \int_A D_{11} L \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \cdot dA \\
 K_{24} &= \int_A \left( \frac{4D_{11}y}{L^2} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) + 2D_{16} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{3}{L} \right) \right) \cdot dA \\
 K_{25} &= \int_A \left( \frac{-8D_{11}y}{L^2} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) + 2D_{16} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 K_{26} &= \int_A \frac{-B_{11}}{L} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \cdot dA & K_{27} &= \int_A D_{11} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \cdot dA \\
 K_{28} &= \int_A \frac{D_{11}}{L} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \cdot dA \\
 K_{29} &= \int_A \left( \frac{4D_{11}y}{L^2} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) + 2D_{16} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA \\
 K_{33} &= \int_A D_{11} L^2 \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right)^2 \cdot dA
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{K}_{34} &= \int_A \left( \frac{4\mathbf{D}_{11}y}{L} \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) + 2\mathbf{D}_{16}L \left( \frac{4x}{L^2} - \frac{3}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{35} &= \int_A \left( \frac{-8\mathbf{D}_{11}y}{L} \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) + 2\mathbf{D}_{16}L \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{36} &= \int_A -\mathbf{B}_{11} \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \cdot dA & \mathbf{K}_{37} &= \int_A \mathbf{D}_{11}L \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \cdot dA \\
 \mathbf{K}_{38} &= \int_A \mathbf{D}_{11}L^2 \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \cdot dA \\
 \mathbf{K}_{39} &= \int_A \left( \frac{4\mathbf{D}_{11}y}{L} \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) + 2\mathbf{D}_{16}L \left( \frac{-4}{L^2} + \frac{6x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{44} &= \int_A \left( \frac{16\mathbf{D}_{11}y^2}{L^4} + 4\mathbf{D}_{66} \left( \frac{4x}{L^2} - \frac{3}{L} \right)^2 + \frac{16\mathbf{D}_{16}y}{L^2} \left( \frac{4x}{L^2} - \frac{3}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{45} &= \int_A \left( \frac{-32\mathbf{D}_{11}y^2}{L^4} + 4\mathbf{D}_{66} \left( \frac{4x}{L^2} - \frac{3}{L} \right) \left( \frac{-8x}{L^2} + \frac{4}{L} \right) - \frac{16\mathbf{D}_{16}y}{L^2} \left( \frac{4x}{L^2} - \frac{3}{L} \right) + \frac{8\mathbf{D}_{16}y}{L^2} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{46} &= \int_A \left( \frac{-4\mathbf{B}_{11}y}{L^3} - \frac{2\mathbf{B}_{16}}{L} \left( \frac{4x}{L^2} - \frac{3}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{47} &= \int_A \left( \frac{4\mathbf{D}_{11}y}{L^2} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) + 2\mathbf{D}_{16} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{3}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{48} &= \int_A \left( \frac{4\mathbf{D}_{11}y}{L} \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) + 2\mathbf{D}_{16}L \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{3}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{49} &= \int_A \left( \frac{16\mathbf{D}_{11}y^2}{L^4} + 4\mathbf{D}_{66} \left( \frac{4x}{L^2} - \frac{3}{L} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right) + \frac{8\mathbf{D}_{16}y}{L^2} \left( \frac{4x}{L^2} - \frac{3}{L} \right) + \frac{8\mathbf{D}_{16}y}{L^2} \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{55} &= \int_A \left( \frac{64\mathbf{D}_{11}y^2}{L^4} + 4\mathbf{D}_{66} \left( \frac{-8x}{L^2} + \frac{4}{L} \right)^2 - \frac{32\mathbf{D}_{16}y}{L^2} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{56} &= \int_A \left( \frac{8\mathbf{B}_{11}y}{L^3} - \frac{2\mathbf{B}_{16}}{L} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{57} &= \int_A \left( \frac{-8\mathbf{D}_{11}y}{L^2} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) + 2\mathbf{D}_{16} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{58} &= \int_A \left( \frac{-8\mathbf{D}_{11}y}{L} \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) + 2\mathbf{D}_{16}L \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{59} &= \int_A \left( \frac{-32\mathbf{D}_{11}y^2}{L^4} + 4\mathbf{D}_{66} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right) + \frac{8\mathbf{D}_{16}y}{L^2} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) - \frac{16\mathbf{D}_{16}y}{L^2} \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA \\
 \mathbf{K}_{66} &= \int_A \frac{\mathbf{A}_{11}}{L^2} \cdot dA & \mathbf{K}_{67} &= \int_A \frac{-\mathbf{B}_{11}}{L} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \cdot dA & \mathbf{K}_{68} &= \int_A -\mathbf{B}_{11} \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \cdot dA
 \end{aligned}$$

$$K_{69} = \int_A \left( \frac{-4B_{11}y}{L^3} - \frac{2B_{16}}{L} \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA$$

$$K_{77} = \int_A D_{11} \left( \frac{-6}{L^2} + \frac{12x}{L^3} \right)^2 \cdot dA$$

$$K_{78} = \int_A D_{11} L \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \cdot dA$$

$$K_{79} = \frac{4D_{11}y}{L^2} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) + 2D_{16} \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right)$$

$$K_{88} = \int_A \frac{D_{11}}{L^2} \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right)^2 \cdot dA$$

$$K_{89} = \int_A \left( \frac{4D_{11}y}{L} \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) + 2D_{16} L \left( \frac{-2}{L^2} + \frac{6x}{L^3} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA$$

$$K_{99} = \int_A \left( \frac{16D_{11}y^2}{L^4} + 4D_{66} \left( \frac{4x}{L^2} - \frac{1}{L} \right)^2 + \frac{16D_{16}y}{L^2} \left( \frac{-8x}{L^2} + \frac{4}{L} \right) \left( \frac{4x}{L^2} - \frac{1}{L} \right) \right) \cdot dA$$

The element stiffness matrix  $[K]_e =$

$$\begin{bmatrix} \frac{A_{11}b}{L} & 0 & \frac{-1 B_{11}b}{2 L} & \frac{-2B_{16}b}{L} & 0 & \frac{-A_{11}b}{L} & 0 & \frac{1 B_{11}b}{2 L} & \frac{2B_{16}b}{L} \\ \frac{12 \cdot D_{11}b}{L^3} & \frac{6 \cdot D_{11}b}{L^2} & \frac{8 \cdot D_{16}b}{L^2} & \frac{-16 \cdot D_{16}b}{L^2} & \frac{-16 \cdot D_{16}b}{L^2} & 0 & \frac{-12 \cdot D_{11}b}{L^3} & \frac{6 \cdot D_{11}b}{L^2} & \frac{8 \cdot D_{16}b}{L^2} \\ & \frac{4 \cdot D_{11}b}{L} & \frac{6 \cdot D_{16}b}{L} & \frac{-8 \cdot D_{16}b}{L^2} & \frac{-8 \cdot D_{16}b}{L^2} & \frac{B_{11}b}{L} & \frac{-6 \cdot D_{11}b}{L^2} & \frac{2 \cdot D_{11}b}{L} & \frac{2 \cdot D_{16}b}{L} \\ & & \frac{28 \cdot D_{66}b}{3 \cdot L} + \frac{4 \cdot D_{11}b^3}{3 \cdot L^3} & \frac{-32 \cdot D_{66}b}{3 \cdot L} - \frac{8 \cdot D_{11}b^3}{3 \cdot L^3} & \frac{-32 \cdot D_{66}b}{3 \cdot L} - \frac{8 \cdot D_{11}b^3}{3 \cdot L^3} & \frac{2B_{16}b}{L} & \frac{-8 \cdot D_{16}b}{L^2} & \frac{2 \cdot D_{16}b}{L} & \frac{4 \cdot D_{66}b}{3 \cdot L} + \frac{4 \cdot D_{11}b^3}{3 \cdot L^3} \\ & & & \frac{64 \cdot D_{66}b}{3 \cdot L} + \frac{16 \cdot D_{11}b^3}{3 \cdot L^3} & \frac{64 \cdot D_{66}b}{3 \cdot L} + \frac{16 \cdot D_{11}b^3}{3 \cdot L^3} & 0 & \frac{16 \cdot D_{16}b}{L^2} & \frac{-8 \cdot D_{16}b}{L} & \frac{-32 \cdot D_{66}b}{3 \cdot L} - \frac{8 \cdot D_{11}b^3}{3 \cdot L^3} \\ & & & & & \frac{A_{11}b}{L} & 0 & \frac{-1 B_{11}b}{2 L} & \frac{-2B_{16}b}{L} \\ & & & & & & \frac{12 \cdot D_{11}b}{L^3} & \frac{-6 \cdot D_{11}b}{L^2} & \frac{-8 \cdot D_{16}b}{L^2} \\ & & & & & & & \frac{12 \cdot D_{11}b}{L} & \frac{6 \cdot D_{16}b}{L} \\ & & & & & & & & \frac{28 \cdot D_{66}b}{3 \cdot L} + \frac{4 \cdot D_{11}b^3}{3 \cdot L^3} \end{bmatrix}$$

Symmetric

The element mass matrix  $[M]_e =$

$$\rho_{\text{comp}} A \cdot h \begin{bmatrix} \frac{1}{3} & \frac{7}{20} & \frac{1}{20}h & \frac{1}{18}a^2 & \frac{1}{9}a^2 & \frac{1}{6} & \frac{3}{20} & \frac{-1}{30}h & 0 \\ & \frac{13}{35} & \frac{11}{210}h & \frac{11}{180}a^2 & \frac{1}{9}a^2 & \frac{1}{20} & \frac{9}{70} & \frac{-13}{420}h & \frac{-1}{180}a^2 \\ & & \frac{1}{105}h^2 & \frac{1}{180}a^2h & \frac{1}{45}a^2h & \frac{1}{30}h & \frac{13}{420}h & \frac{-1}{140}h^2 & 0 \\ & & & \frac{2}{75}a^4 & \frac{1}{75}a^4 & 0 & \frac{-1}{180}a^2 & 0 & \frac{-1}{150}a^4 \\ & & & & \frac{8}{75}a^4 & \frac{1}{9}a^2 & \frac{1}{9}a^2 & \frac{-1}{45}a^2L & \frac{1}{75}a^4 \\ & & & & & \frac{1}{3} & \frac{7}{20} & \frac{-1}{20}h & \frac{1}{18}a^2 \\ & & & & & & \frac{13}{35} & \frac{-11}{210}h & \frac{11}{180}a^2 \\ & & & & & & & \frac{1}{105}h^2 & \frac{-1}{180}a^2h \\ & & & & & & & & \frac{2}{75}a^4 \end{bmatrix}$$

symmetric

The element load vector  $[F]_e$  ;

$$[F]_e = \{\delta q_i\} \cdot \left( \int_0^h P \cdot f_{\text{axial}} \cdot dx + \int_0^h \Omega \cdot f_{\text{bend}} \cdot dx + \int_0^h T \cdot f_{\text{torsion}} \cdot dx \right)$$

$$[F]_e = P \cdot \begin{bmatrix} \frac{h}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{h}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \Omega \cdot \begin{bmatrix} 0 \\ \frac{1}{2}h \\ \frac{1}{12}h^2 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}h \\ \frac{1}{-12}h^2 \\ 0 \end{bmatrix} + T \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{h}{6} \\ \frac{2h}{3} \\ 0 \\ 0 \\ \frac{h}{6} \end{bmatrix} = \begin{bmatrix} P \frac{h}{2} \\ \frac{1}{2}\Omega \cdot h \\ \frac{1}{12}\Omega \cdot h^2 \\ \frac{T \cdot h}{6} \\ \frac{2T \cdot h}{3} \\ P \frac{h}{2} \\ \frac{1}{2}\Omega \cdot h \\ \frac{-1}{12}\Omega \cdot h^2 \\ \frac{T \cdot h}{6} \end{bmatrix}$$