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QUALITATIVE BEHAVIOR OF EIGENFREQUENCIES FOR FREE VIBRATION OF AEROSPACE VEHICLE APPENDAGES

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ABSTRACT

In this work, the changes in eigenfrequencies of flexible appendages with various mass ratios of space vehicles due to the allowance of a rigid body motion are studied. These changes are governed by the difference eigenvalue problem [2] which is formulated in terms of the generated effective mass and stiffness matrices. The computation techniques for calculating the upper and lower bounds of the dominant eigenfrequencies are performed by the utilization of the bound formula approach [1] . An example of the transverse vibration analysis of the fuslage-wing combination with different mass ratios in different zones of vibratory motion is solved . A comparison between the computed calculations and previously published results [10] is presented

INTRODUCTION

In this paper an analysis is presented for the qualitative behavior of the deformed mode eigenfrequencies for small free vibrations about configurations of stable equilibrium for aerospace vehicle appendages, including major structural components. The deformed mode eigenfreuencies of a structural system moving freely in space undergo changes in attitude because of the variation in kinetic and potential energies assoicated with rigid body and deformed motions. It is shown that allowing rigid body motion results in an apparent decrease in the inertia characteristics of the system causing an increase (or no change) in the deformed mode eigenfrequencies [2] . Moreover restricting elastic deformation modes of motion by imposing redundant con-constraints results in an increase in the stiffness characteristics of the system, also causing an increase (or no change) in the deformed mode eigenfrequencies [6] . In the limiting case, called herein the "stationarity state," the deformed mode eigenfrequencies tend to their lowest possible values. This state arises for a structural system which neither performs rigid body motion nor possesses redundant constraints. The former state is denoted herein as the "underconstrained zone" and the latter state as the "overconstrained zone". The method presented has the capability of specifying the boundaries for the variations of all natural frequencies for any elastic component or appendage. This

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in turn identifies their respective resonance spectrums. Although the example presented to illustrate the general method is restricted to flexure, the procedure may be readily applied to complex systems involving any possible combination of axial, torsional and flexural degrees of system.

This work is an attempt to help the analyst and designer gain a better physical understanding of the mutual effect of either underconstrained or overconstrained motion for any particular appendage or component of an aerospace vehicle. In the "underconstrained zone' the variations in the natural frequencies can be controlled by altering the effective inertia characteristics of the particular component. Moreover in the overconstrained zone an alteration of the effective stiffness characteristics plays the same role. The method is applicable throughout the entire range of ratios of elastic appendage mass to the total mass of the system. This meets the requirements for analyzing aerospace vehicles with every massive appendages such as those currently being proposed for future design applications.

EQUATIONS OF MOTION AND EIGENVALUE PROBLEM

Assuming that a given physical body is idealized as a discrete system, one can write the kinetic and potential energies respectively in matrix form

$$2T = \dot{\tilde{q}}^T \tilde{m}_{qq} \dot{\tilde{q}} + 2\dot{\tilde{q}}^T \tilde{m}_{qp} \dot{\tilde{p}} + \dot{\tilde{p}}^T \tilde{m}_{pp} \dot{\tilde{p}} ,$$

$$2V = \tilde{p}^T \tilde{k}_{pp} \tilde{p}, \tag{1}$$

where $\tilde{q}$ and $\tilde{p}$ play the role of the generalized coordinate vectors for rigid and deformed modes of motion respectively. Applying Lagrang's equations to the expression given by equation (1), and solving the resulting equations for q_i in terms of the p_i coordinates, one obtains

$$(\tilde{m}_{pp} - \tilde{m}_{ef})\ddot{\tilde{p}} + \tilde{k}_{pp}\tilde{p} = \tilde{Q} \cdot \tag{2}$$

It may be written more simply as

$$\tilde{m}^* \ddot{\tilde{p}} + \tilde{k}\,\tilde{p} = \tilde{Q} , \tag{3}$$

where

$\tilde{m}_{pp}$ and $\tilde{k}_{pp}(=\tilde{k})$ are the mass and stiffness matrices of the system measured in the p-coordinate system respectively,

$\tilde{m}^*$ is the reduced mass matrix given by

$$\tilde{m}^* = \tilde{m}_{pp} - \tilde{m}_{ef} = \tilde{m}^{*T},$$

$\tilde{m}_{ef}$ is the effective mass matrix given by

$$\tilde{m}_{ef} = \tilde{m}_{qp}^T \tilde{m}_{qq}^{-1} \tilde{m}_{qp} .$$

Note that the $\tilde{m}_{ef}$ effective mass matrix results from allowing rigid body motion of the system, such as translatory (t), rotary (r) or a combination of these motions (tr). Therefore the expressions given in equation (1) express the kinetic and potential energies of the system in the "underconstrained zone." In view of equation (3), equation (1) may then be recast as

$$T = \frac{1}{2}\,\dot{\boldsymbol{p}}^T \boldsymbol{m}^* \dot{\boldsymbol{p}} \quad ,$$

$$V = \frac{1}{2}\,\boldsymbol{p}^T \boldsymbol{k}\,\boldsymbol{p} \quad . \tag{4}$$

In the "underconstrained zone", the kinetic energy for a given configuration of the system is therefore decreased because of rigid body motion, while the potential energy is unaltered [2] . Hereby the deformed mode natural frequencies cannot be less than the respective frequencies in the "stationarity zone". If the system under consideration is subjected to constraints which imply that (1) the rigid body motions are eliminated and (2) there are no differences between the number of dynamic and static degrees of freedom of the system then the $\boldsymbol{m}_{ef}$ effective mass disappears and equation (3) becomes

$$\boldsymbol{m}\,\ddot{\boldsymbol{p}} + \boldsymbol{k}\,\boldsymbol{p} = \boldsymbol{0} \quad . \tag{5}$$

where $\boldsymbol{m}$ and $\boldsymbol{k}$ play the role of $\boldsymbol{m}_{pp}$ and $\boldsymbol{k}_{pp}$ in equation (1). Equation (5) is therefore the governing equation of motion of the physical body in the "stationarity state." Next, in order to simplify the concept, let us consider the $\boldsymbol{p}$ generalized coordinate vector comprised of a dynamic vector $\boldsymbol{p}^*$ and an excess+ static one $\boldsymbol{p}_s$. Usually more static degrees of freedom are used for a more accurate description of the system's elastic properties [4,7] . Since by definition $\dot{\boldsymbol{p}}_s = 0$, the kinetic and potential energies in the stationarity state may be written as

$$T = \frac{1}{2}\,\dot{\boldsymbol{p}}^{*T} \bar{\boldsymbol{m}}\,\dot{\boldsymbol{p}}^* \quad ,$$

$$V = \frac{1}{2}\,\boldsymbol{p}^{*T} \boldsymbol{k}^* \boldsymbol{p}^* + \boldsymbol{p}_s^T \boldsymbol{k}_s \boldsymbol{p}^* + \frac{1}{2}\,\boldsymbol{p}_s^T \boldsymbol{k}_{ss} \boldsymbol{p}_s = \frac{1}{2}\,\boldsymbol{p}^T \boldsymbol{k}\,\boldsymbol{p} \tag{6}$$

where the mass matrix $\bar{\boldsymbol{m}}$ is a condensed form of $\boldsymbol{m}$, noting that the kinetic energy is invarient. The stiffness matrix measured in the new coordinate system (p_i^*, p_{si}) may be partitioned as

$$\boldsymbol{k} = \begin{bmatrix} \boldsymbol{k}^* & \boldsymbol{k}_s^T \\ \boldsymbol{k}_s & \boldsymbol{k}_{ss} \end{bmatrix} \qquad \boldsymbol{p} = \begin{Bmatrix} \boldsymbol{p}^* \\ \boldsymbol{p}_s \end{Bmatrix}$$

Applying Lagrange's equations to equation (6) and assuming that p_i^* play the role of master coordinates, while p_{si} play the role of slave coordinates [5] , one obtains the alternate form of equation (5)

$$\bar{\boldsymbol{m}}\,\ddot{\boldsymbol{p}}^* + (\boldsymbol{k}^* - \boldsymbol{k}_{ef})\boldsymbol{p}^* = \boldsymbol{0} \quad , \tag{7}$$

where the effective stiffness matrix $\boldsymbol{k}_{ef} = \boldsymbol{k}_s^T \boldsymbol{k}_{ss}^{-1} \boldsymbol{k}_s$. Now, let us impose s constraints to eliminate the excess static degrees of freedom. The equation of motion of the overconstrained system is then given by

$$\bar{\bar{\boldsymbol{m}}}\,\ddot{\boldsymbol{p}}^* + \boldsymbol{k}^* \boldsymbol{p}^* = \boldsymbol{0} \quad , \tag{8}$$

+excess static degrees of freedom are those in excess of the assigned where $\tilde{k}^*$ represents the stiffness matrix of the overconstrained system. The inverse is also true. The effective stiffness matrix begins to appear in equation (8) by the addition of the static degrees of freedom to the overconstrained system and in the limit equation (8) takes the form of equation (7). The comparison of equations (7) and (8) shows that the $\tilde{k}_{ef}$ effective stiffness matrix depends on the deformed body motion of the system. Therefore the potential energy of a given configuration is increased from its stationarity state by the addition of redundant constraints while its kinetic energy for the given motion is unaltered. The deformed mode natural frequency cannot be less than the respective frequency in the "stationarity state". Moreover, in general the addition of redundant constraints to the system in the "stationarity state", tends therefore to increase the potential energy. From this viewpoint equation (8) is considered the governing equation of motion for the physical system in the "overconstrained zone. In summary, equation (5) may be viewed as the governing equation of motion of a physical body through a limiting process of (1) "underconstrained zone" where the deformed mode eigenfrequencies approach the stationarity zone from above by eliminating rigid body motion and (2) the "overconstrained zone" where the eigenfrequencies also approach the stationarity state from above by eliminating redundant constraints.

In each zone of vibratory motion for space vehicle appendages and major components, the determination of the deformed mode eigenfrequencies requires the reduction of equations (3), (5) and (8) to the characteristic eigenvalue problem. Since the lower mode dominate the response, one may reasonably ignore oscillations in higher mode [8] , and for that the eigenvalue problem may be arranged in the form

$$\tilde{D}\tilde{p} = \lambda \tilde{p}. \tag{9}$$

In such cases $\lambda = \omega^{-2}$, where ω is the natural frequency and the dynamic matrix $\tilde{D}$ is the product of the flexibility and mass matrices given by

$$\tilde{D} = \tilde{D}_f = \tilde{k}^{-1}(\tilde{m}-\tilde{m}_{ef}), \text{ for the "underconstrained zone"} \tag{10}$$

$$\tilde{D} = \tilde{D}_c = (\tilde{k}-\tilde{k}_{ef})^{-1}\tilde{m}, \text{ for the "overconstrained zone"} \tag{11}$$

$$\tilde{D} = \tilde{D}_o = \tilde{k}^{-1}\tilde{m}, \text{ for the "stationarity state".} \tag{12}$$

Note that $\tilde{D}_f = \tilde{D}_o - \tilde{D}_{ef} = \tilde{k}^{-1}\tilde{m} - \tilde{k}^{-1}\tilde{m}_{ef}$. (13)

However, if the inverse dynamic matrix is required for any case, the form

$$\tilde{E}_c = \tilde{E}_o - \tilde{E}_{ef} = \tilde{m}^{-1}\tilde{k} - \tilde{m}^{-1}\tilde{k}_{ef}$$

One can now examine the qualitative behavior of deformed mode eigenfrequencies for small free vibration of a space vehicle model or any of its structural components about a stable equilibrium configuration by this procedure. To this end the following conclusion is relevant. The deformed mode natural frequencies of free vibrating system are lower in the "stationarity state" defined by equation (12) than in either in the "underconstrained zone" defined by equation (10) or in the "overconstrained zone" defined by equation (11), as shown in Figures 3 and 4. Mathematically this may be written as

$$\omega_{fi} \geq \omega_{oi} \leq \omega_{ci}, \quad \text{for} \quad i = 1,2,3,... \tag{14}$$

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EIGENANALYSIS PROCEDURE

Eigenanalysis of the plane transverse vibrations of the fuselage-wing combination of reference [10] is presented in order to demonstrate the application of the present procedure. The single element idealization of a uniform wing attached to a fuselage mass is shown in Figure 1. For the unification and comparison of results, the assumptions, notation and data given on page 328, reference [10] are used. The element mass and stiffness matrices measured in the local coordinate system may then be partitioned as

$$\underset{\sim}{M} = \begin{bmatrix} \underset{\sim}{m}_{I} & \underset{\sim}{m}_{I\ II} \\ \underset{\sim}{m}^{T}_{I\ II} & \underset{\sim}{m}_{II\ II} \end{bmatrix} \qquad \underset{\sim}{K} = \begin{bmatrix} \underset{\sim}{k}_{I\ I} & k_{I\ II} \\ \underset{\sim}{k}^{T}_{I\ II} & \underset{\sim}{k}_{II\ II} \end{bmatrix}$$

Note that the ratio of the fuselage mass to the wing mass is denoted hereafter as $R = m_F/m_W$.

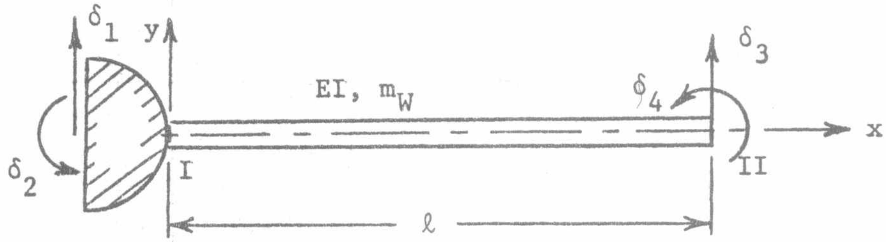

Figure 1 Two node beam element idealization of a uniform wing attached to fuselage mass, where EI = flexural rigidity, ℓ = half span length, $2m_W$ = wing mass and $2m_F$ = fuselage mass.

The mass and stiffness submatrices measured in the generalized coordinate system can then be obtained by using the transformation equation

$$\begin{Bmatrix} \underset{\sim}{q} \\ \underset{\sim}{p} \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{U} & \underset{\sim}{O} \\ \underset{\sim}{C} & \underset{\sim}{U} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\delta}_{I} \\ \underset{\sim}{\delta}_{II} \end{Bmatrix}$$

Therefore

$$\underset{\sim}{m}_{qq} = \underset{\sim}{m}_{I\ I} + \underset{\sim}{C}^{T}\underset{\sim}{m}_{II\ II}\underset{\sim}{C} - \underset{\sim}{m}_{I\ II}\underset{\sim}{C} - \underset{\sim}{C}^{T}\underset{\sim}{m}^{T}_{I\ II} ,$$

$$\underset{\sim}{m}_{qp} = \underset{\sim}{m}_{I\ II} - \underset{\sim}{C}^{T}\underset{\sim}{m}_{II\ II} ,$$

$$\underset{\sim}{m} = \underset{\sim}{m}_{pp} = \underset{\sim}{m}_{II\ II} ,$$

$$\underset{\sim}{k} = \underset{\sim}{k}_{pp} = \underset{\sim}{k}_{II\ II} ,$$

where $\underset{\sim}{U}$ is a unit matrix and $\underset{\sim}{C}$ is a transformation matrix derived from the geometric conditions as shown in Figure 2. The suffix I and II denote the node number. The results of the previous section have been evaluated numerically for various cases of free flexural vibration of the fuselage-wing combination as illustrated in Figure 2. There is a wide variety of numerical methods available for solving the standard eigenvalue problem given by equation (12), [9] . The bound formula [1] will be selected for the computation of upper and lower bounds for the fundamental frequency of a space vehicle model for a variety of cases, applying the simplest form

$$(\mathrm{Tr}\underset{\sim}{D})^{-1/2} \leq \omega \leq (\mathrm{Tr}\underset{\sim}{D}^2/\mathrm{Tr}\underset{\sim}{D})^{-1/2} \qquad (15)$$

In view of equations (9) and (12) the fundamental frquency at the "stationrity state" is then bounded by

$$3{,}514675\sqrt{EI/m_w \ell^3} \leq \omega_o \leq 3{,}532639\sqrt{EI/m_W \ell^3}$$

where the exact the value [3] is equal to 3,516 $\sqrt{EI/m_W \ell^3}$ sec^{-1}. For the sake of comparison with the numerical results of reference [10], (pp. 331 and 333), the bounds of the nondimensional fundamental frequency $\tilde{\omega}$ were computed for the cases 1.2 and 1.3 in Figure 2 for the mass ratios R = O, 1 and 3. The results are shown in Figure 3 and also in Table 1.

R	Computed Values	Results of [9]
O	$5{,}561 \leq \tilde{\omega}_{(f,t)} \leq 5{,}651$	5,606
1	$4{,}202 \leq \tilde{\omega}_{(f,t)} \leq 4{,}257$	4,229
3	$3{,}812 \leq \tilde{\omega}_{(f,t)} \leq 3{,}857$	3,835
*	$17{,}019 \leq \tilde{\omega}_{(f,r)} \leq 18{,}050$	17,544

Table 1 Nondimensional fundamental frequency $\tilde{\omega} = \omega/\sqrt{EI/m_W \ell^3}$ for the uniform wing attached to a fuselage mass performing translatory (f,t) and rotary (f,r) rigid body motion, (* rotary inertia of the fuselage is neglable).

For a complete analysis, the bounds of the fundamental frequency ratio $T = \omega/\omega_o$ for the cases shown in Figure 2 were computed for various values of mass ratio. The average value of each bound pair is then listed in Table 2 and also shown in Figure 3.

R	O	1	2	3	4	5	10	∞
$T_{(f,tr)}$	7,5876	5,3458	5,1967	5,1425	5,1144	5,0972	5,0622	4,9636
$T_{(f,t)}$	1,5868	1,1972	1,1191	1,0854	1,0667	1,0546	1,0287	1

Table 2 The fundamental frequency ratio T of the wing fuselage combination performing rigid plane motion (f,tr) and translatory rigid motion (f,t) for various value of mass ratio.

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The results of the fundamental frequency ratio for the remaining cases which are independent of the mass ratio are as follows

$$T_{(f,r)} = 4,9636 \ , \quad T_{(c,t)} = 1,7491 \ , \quad T_{(c,r)} = 5,8012$$

A review of the results listed in Tables 1 and 2 and also shown in Figure 3 indicates the following remarks

1. The lowest "stationarity" values of the deformed natural frequencies are reached for systems performing neither rigid body motion nor possessing redundant constraints, regardless of the values of mass ratios.
2. In general the natural frequencies decrease (or no change) for an increase in the mass ratio concerned with any component of the aerospace vehicle as illustrated in Figure 4.
3. The case of a vanishingly small rotational inertia of the fuselage leads to an invariance in the natural frequencies for various mass ratios (i.e. point Q in Figure 4-b) due to rotary rigid body motion. In the limit when the mass ratio approaches infinity (i.e. boundary PQR), there is a sudden change in natural frequencies as indicated by the boundary of the variation region shown in Figure 4-b. A similar situation arises for the translatory rigid body motion of a system with negliable mass and significant mass moment of inertia, such as an antenna.
4. If the mass moment of inertia of any appendage is considerable then the concurrent point Q disappears. However the effects of rotary rigid body motion on the variations of natural frequencies increase as the mass moment of inertia increases. In this case the nature of the changes in natural frequencies would be similar to that of the translatry rigid body motion shown in Figures 3 and 4.
5. In particular, for the mass ratio R = O, there are appreciable changes in the natural frequencies. However this particular case conforms physically with the analysis of the vehicle as a whole. When the mass of the fuselage is significantly greater than the mass of the appendage in question, the variations in natural frequencies of the appendage have a nearly linear behavior and in the limit they tend to be invariant (as in the case of the overconstrained zone).
6. The governing equality (14) permits one to conclude that the region of variation for each deformed mode natural frequency is similar to that of the fundamental frequency for various mass ratios as illustrated in Figure 4. More investigation into the behavior of the system (particularly in the overconstrained zone), may be easily performed by applying the present procedure to a mathematical model involving higher degrees of freedom. The information included in Figures 3 and 4 efficiently specifies the resonance zone for each component of the aerospace vehicle.

CONCLUSION

The proposed procedure provides appreciable eigeninformation for various components of aerospace vehicles under axial, torsional, flexural or combined vibratory motion. The procedure is a method of analysis applicable to a wide variety of related practical problems. The results of the elementary illustrative example clearly demonstrate that the proposed method is reliable for eigenanalysis of unconstrained systems, such as the wing-fuselage combination analyzed herein for a variety of rigid and deformed body motions. It is also uesful for specifying the variation boundaries of the deformed mode natural frequencies for various appendage mass ratios and for various types of rigid body motion as illustrated in Figure 3 and 4. This information, when used in conjunction with the resonance spectrum for various components of an aerospace vehicle, has considerable value to the design engineer.

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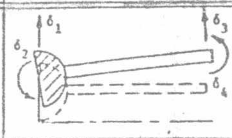
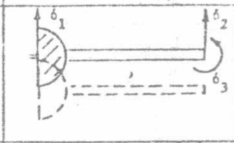
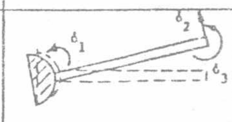
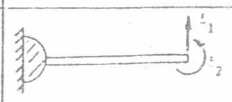
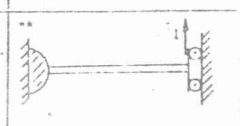
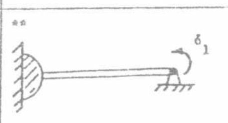
*State of a fuselage-wing combination (A,B)	Mathematical model	q vector	p vector
1.1 Plane rigid body motion (f,tr)		$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}$	$\begin{bmatrix}\delta_3-\delta_1-\ell\delta_2\\ \delta_4-\delta_2\end{bmatrix}$
1.2 Translatory rigid body motion (f,t)		δ_1	$\begin{bmatrix}\delta_2-\delta_1\\ \delta_3\end{bmatrix}$
1.3 Rotary rigid body motion (f,r)		δ_1	$\begin{bmatrix}\delta_2-\ell\delta_1\\ \delta_3-\delta_2\end{bmatrix}$
2.0 Stationarity state. No rigid body motion, no redundant constraints		—	$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}$
3.1 Translatory deformed body motion, one redundant constraint (c,t)	** 	—	δ_1
3.2 Rotary deformed body motion, one redundant constraint (c,r)	** 	—	δ_1

Figure 2 Sketch of Configurations for Fuselage-Wing Combinations in Various Zones of Flexural Vibratory Motion.

*The index (A) denotes the system either within the underconstrained (f) or overconstrained (c) zones. The index (B) describes the type of performance as plane (tr), translatory (t) or rotary (r) motion.

**Overconstrained motion which may arise in a ground test or in the case of a very high mass ratio $R = m_F/m_W$.

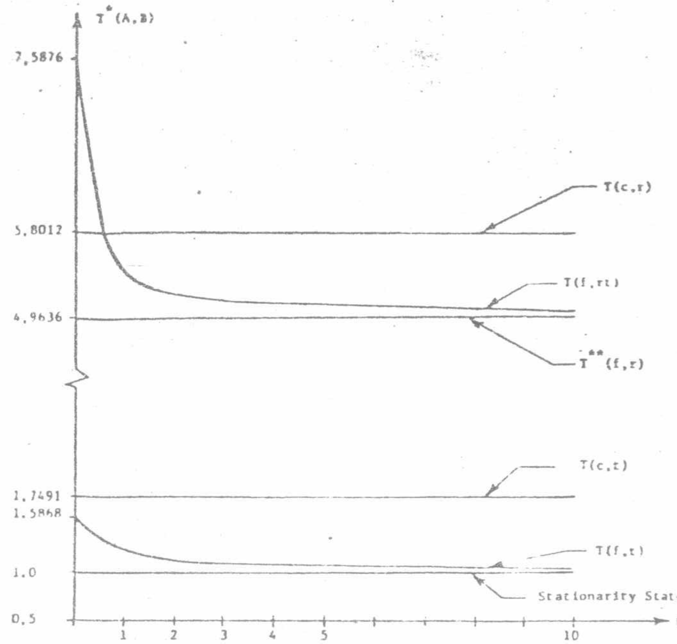


Figure 3 Graph of Fundamental Frequency Ratio T Versus Mass Ratio R for Fuselage-Wing Combinations

*The index (A) describes the nature of the constraint. The index (B) describes the type of motion.

**Rotary inertia of the fuselage is negligible

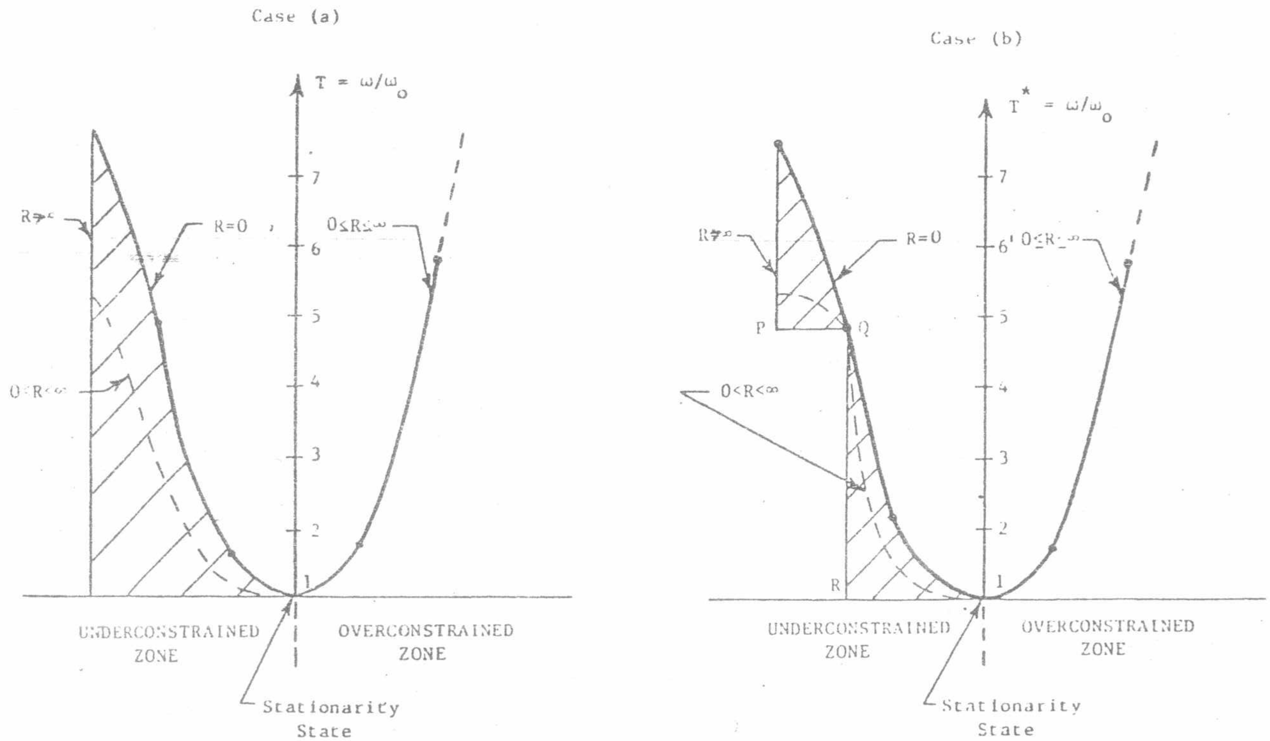


Figure 4. Typical Sketches of the Regions of Variations in the Fundamental Frequency Ratio T Versus the Mass Ratio R for the Fuselage-Wing Combination in the Various Constrained Zones

*For negligible rotary inertia of the fuselage, the region of variation T (dashed area) is more closely bounded as shown in case (b) compared to case (a).