MILITARY TECHNICAL COLLEGE

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ANALYTICAL INVESTIGATION OF ELASTIC MECHANISMS

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ABSTRACT

To achieve the performance of mechanisms and machines to a higher accuracy, the elastic deformations of their members, under dynamic conditions, are to be taken into account. In the presented work each member in the linkage is treated as being elastic. The kinematic equations of constraints at the kinematic pairs are presneted. These equations are introduced into the equations of motion through Lagrange multipliers, to obtain a coupled system of nonlinear algebric and differential equations. The later system will have the same form for each element. So, the total set of equations, describing the linkage, can be generated automatically. Illustrative examples are given and the agreement of the results with those previously solved using other techniques is satisfactory.

INTRODUCTION

During last decade, several researchs [1-9] have introduced improved mathematical models of planar systems taking into account the elastic behaviour of their members. The stiffness matrix approach of structural analysis was used by Winfrey [1] . The technique developed in [2] was based on flexibility matrix approach. Recently, the finite element technique is used. [3-5] for vibration analysis of general planar mechanisms. The effects of both coriolis accelerations and centrifugal forces, arising due to elastic deformations, are discussed and obtained in the governing equations of motion in [6] .

Due to the fact that the mechanism characteristics change with its position, most researchs use the concept of instantaneous structure [1-6] . However, a sensitivity approach is introduced in [3] premitting the calculation of eigensolutions as a function of the mechanism position. A few experimental work [6-8] have been carried out to verify the theoritical investigations.

In this work, a sofisticated technique is proposed for the analysis of planar mechanisms composed of elastic members. The compatability conditions at joints and the equations of motion are coupled to obtain a system of differential and algebraic equations. The presented formulation can be applied to any planar linkage, Fig.1, having turning and/or sliding

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kinematic pairs. The degree of freedom of its rigid body motion is one.

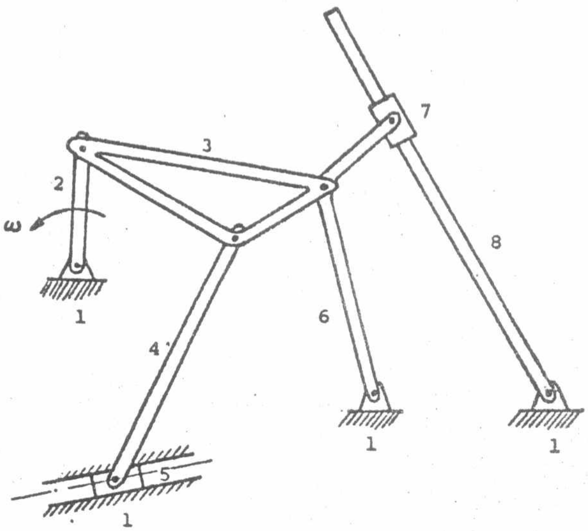

Fig.1. General Planar Mechanism.

GENERAL BEAM ELEMENT

A general elastic beam element, moving in a plane, is shown in Fig.2, in two frames of references. The inertial (x,y) and the local-body fixed ($\bar{x},\bar{y}$) frames. The later frame being such that its origin coincides with the element mass center C. The position vector $\vec{R}_C$ and the angle of rotation θ define the position of the local frame with respect to inertia one. They represent the rigid body motion as well as the elastic deformations at C.

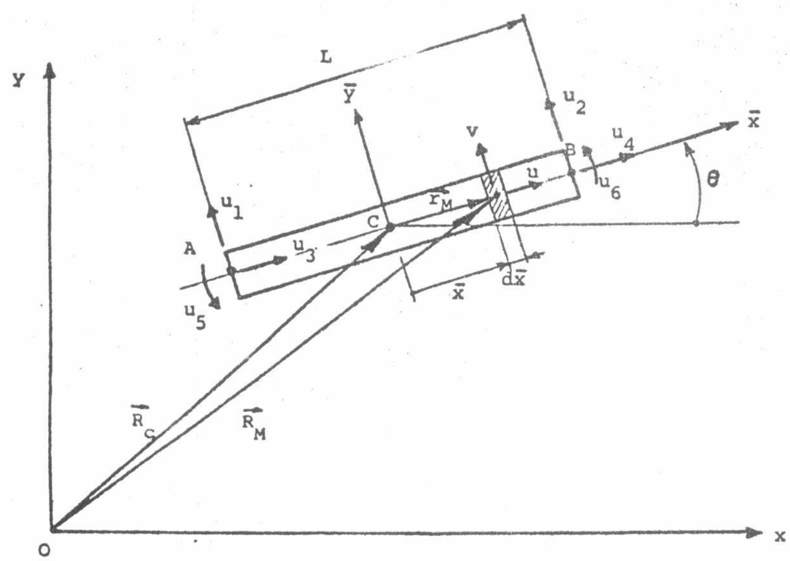

Fig.2. General Beam Element.

Let u,v be the components of elastic deformations of any point $M(\bar{x},0)$, measured relative to local system. These deformations can be written in terms of relative nodal displacements u_j, j=1,2,...,6, by introducing shape functions. Thus, it is possible to locate the position of all points of the element by specifying the coordinates (x,y) of C , the angle θ and the six relative nodal displacements u_j. These nine generalized coordinates are not all independent, three of them can be eliminated using boundary conditions.

The position vector of point M, in its deformed position, is given by

$$\vec{R}_M = \vec{R}_c + \vec{r}_M$$

Or, in matrix form

$$\begin{bmatrix} x_M \\ y_M \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + R(\theta) \begin{bmatrix} \bar{x}+u \\ v \end{bmatrix} \quad (1)$$

where, the rotation matrix $R(\theta)$ is given by

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The elastic deformations u and v are expressed in terms of nodal displacements as [10]

$$u = \phi_3 u_3$$
$$v = \phi_1 u_1 + \phi_2 u_2 \quad (2)$$

where the shape functions ϕ_j, j =1,2,3 are

$$\phi_1 = 2\xi^2 - 4\xi^3$$
$$\phi_2 = 2\xi^2 + 4\xi^3 \quad (3)$$
$$\phi_3 = -2\xi$$

with $\xi = \bar{x}/L$ is a nondimensional parameter, and L is the element length. It should be mentioned that u,v are measured relative to the deformations at C. So, in addition to 6 boundary conditions at nodal points, one have at C

$$u(0) = v(0) = v'(0) = 0$$

where $|'|$ indicates differentiation with respect to $\bar{x}$.

Kinetic and Potential Energies

The angular velocity, with respect to inertial frame, of any differential segment (Fig.2) on the center line of the element is given by

$$\dot{\theta} + \dot{v}'$$

where $|\dot{}|$ indicates differentiation with respect to time.

The kinetic energy of the element is written as

$$T = \frac{1}{2} \int_{-L/2}^{L/2} \left[m_{\bar{x}} (\dot{x}_M^2 + \dot{y}_M^2) + J_{\bar{x}} (\dot{\theta} + \dot{v}')^2 \right] d\bar{x}$$

where, $m_{\bar{x}}$ is the mass per unit length and $J_{\bar{x}}$ is the mass moment of inertia of the segment about an axis perpendicular to the plane of motion. Assuming that $m_{\bar{x}}$ and $J_{\bar{x}}$ are constants, then

$$T = \frac{M}{2} \int_{-1/2}^{1/2} (\dot{x}_M^2 + \dot{y}_M^2) d\xi + \frac{MI}{2A} \int_{-1/2}^{1/2} (\dot{\theta} + \frac{1}{L} \frac{\partial \dot{v}}{\partial \xi})^2 d\xi \qquad (4)$$

where M, I and A are the total mass, the area moment of inertia about neutral axis and the cross sectional area of the element, respectively. Using Eqns (1), (2) and (3), the kinetic energy, in terms of generalized displacements and velocities will be

$$\begin{aligned} T = \frac{M}{2} \Big[& \dot{x}^2 + \dot{y}^2 + \frac{1}{12} L^2 \dot{\theta}^2 + \frac{1}{3} (L\dot{\theta}^2 u_3 + \dot{\theta}^2 u_3^2 + \dot{u}_3^2) \\ & - \frac{1}{3} \dot{\theta} (\dot{x} \cos\theta + \dot{y} \sin\theta)(u_1 + u_2) + \frac{1}{5} \dot{\theta}^2 (u_1 + u_2)^2 \\ & - \frac{1}{3} (\dot{x} \sin\theta - \dot{y} \cos\theta)(\dot{u}_1 + \dot{u}_2) + \frac{1}{5} (\dot{u}_1 + \dot{u}_2)^2 \\ & - \frac{1}{5} \dot{\theta} \dot{u}_3 (u_1 - u_2) + \frac{1}{28} \dot{\theta}^2 (u_1 - u_2)^2 \\ & + \frac{1}{10} \dot{\theta} (2u_3 - L)(\dot{u}_1 - \dot{u}_2) + \frac{1}{28} (\dot{u}_1 - \dot{u}_2)^2 \Big] \\ + \frac{MI}{2A} \Big[& \dot{\theta}^2 - \frac{2}{L} \dot{\theta} (\dot{u}_1 - \dot{u}_2) + \frac{4}{3L^2} (\dot{u}_1 + \dot{u}_2)^2 + \frac{9}{5L^2} (\dot{u}_1 - \dot{u}_2)^2 \Big] \end{aligned} \qquad (5)$$

If a slider exists at one of the extremities of an element, it may be entered into the expression of T as a concentrated mass.

The potential energy of the general elastic beam can be obtained, using $Eq^{n}s$ (1),(2) and (3), in terms of generalized displacements as follows

$$V = \frac{32\,EI}{L^3} \left[u_1^2 - u_1 u_2 + u_2^2 \right] + \frac{2EA}{L} (u_3^2) \qquad (6)$$

where E is Young's modulus, assumed to be constant.

Generalized Forces

The generalized forces Q_j, j=1,2,...9, associated with the corresponding generalized coordinates, are obtained from the expression of virtual work as follows

$$W = Q_x \delta x + Q_y \delta y + Q_\theta \delta\theta + \sum_{j=1}^{6} Q_j \delta u_j \qquad (7)$$

COMPATABILITY CONDITIONS AT JOINTS

The chosen generalized coordinates are related to those of the adjacent element (s) through the equations of constraint. These equations are kinematic in nature and specify the compatability conditions of elastic deformations at the kinematic pair (K.P) between two elements.

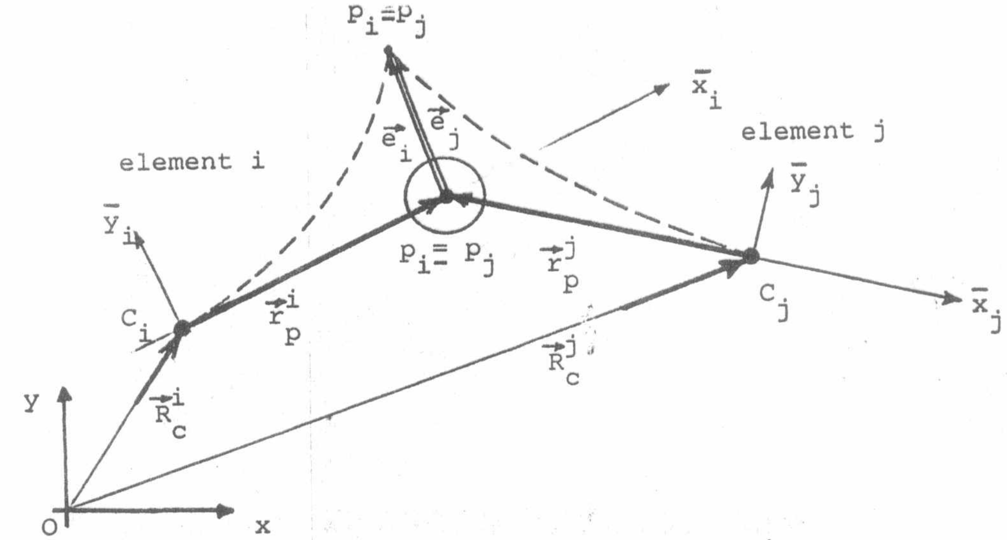

Fig.3. Turning Kinematic Pair.

Fig.3, represents a turning K.P between elements i and j with mass centers C_i and C_j, respectively. Two points $p_i(\bar{x}_i,0)$, on element i, and $p_j(\bar{x}_j,0)$, on element j, are located before deformations, by $\vec{r}^i_p$ and $\vec{r}^j_p$, respectively. The elastic deformations of p_i and p_j are specified by the vectors $\vec{e}_i$ and $\vec{e}_j$. From Fig.3, one can obtain the following vectorial equation

$$\vec{R}^i_c + \vec{r}^i_p + \vec{e}_i - \vec{e}_j - \vec{r}^j_p - \vec{R}^j_c = 0 \qquad (8)$$

or, in components, with respect to inertial frame, one have

$$x_i + (\bar{x}_i + u_i)\cos\theta_i - v_i \sin\theta_i - x_j - (\bar{x}_j + u_j)\cos\theta_j + v_j \sin\theta_j = 0 \qquad (9)$$

$$y_i + (\bar{x}_i + u_i)\sin\theta_i + v_i \cos\theta_i - y_j - (\bar{x}_j + u_j)\sin\theta_j - v_j \cos\theta_j = 0 \qquad (10)$$

The equations (9) and (10) imply that the points p_i and p_j have the same deflections in any two perpendicular directions.

Fig.4, represents a sliding K.P where the slider is attached to element j and considered to be a concentrated mass. The points p_i and p_j coincide after elastic deformations. Eq. (8) remains applicable in this case. However, $(\vec{r}^j_p)^\circ \neq 0$, $(\vec{r}^i_p)^\circ = 0$ in case of sliding joint, while $(\vec{r}^j_p)^\circ = (\vec{r}^i_p)^\circ = 0$ in case of turning joint. So, a relative displacement between the two elements is allowed at the slider. Another difference between the two joints is that in sliding joint the deflections of p_i and p_j in direction of $\bar{y}_i$ are the same, while their deflections in $\bar{x}_i$ direction are completely independent. There is only one constraint equation for the sliding joint and may be obtained by defining the component of Eq.(8) in $\bar{y}_i$ direction as follows

$$x_i \sin\theta_i - y_i \cos\theta_i - x_j \sin\theta_i - y_j \cos\theta_i - \bar{x}_j \sin(\theta_i-\theta_j)$$
$$- v_i - u_j \sin(\theta_i-\theta_j) + v_j \cos(\theta_i-\theta_j) = 0 \qquad (11)$$

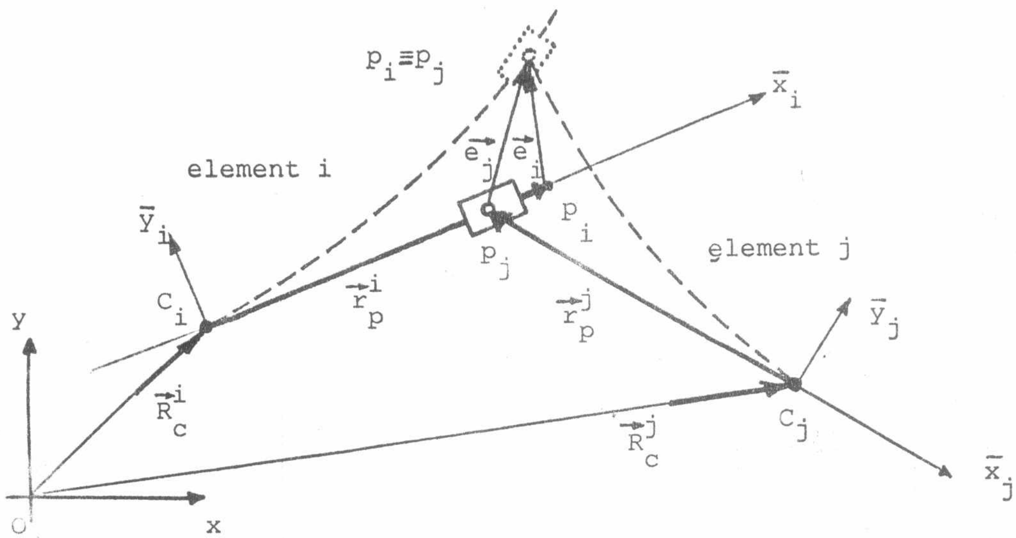

Fig.4. Sliding Kinematic Pair.

Any member in the linkage may be divided into several elements. In this case, in addition to equations (9) and (10) one must have

$$\theta_i + v'_i - \theta_j - v'_j = 0 \qquad (12)$$

Eq. (12) implies that the orientation of the deformed $i^{\underline{th}}$ element at point p_i, relative to inertial frame, equals that of the $j^{\underline{th}}$ element at point p_j. Any equation of constraint between the elements i and j may be written in the form

$$\phi^{ij}(q^i, q^j) = 0 \qquad i < j \qquad (13)$$

where the vector $q^n = [x_n \; y_n \; \theta_n \; u_1^n \; ..... \; u_6^n]^t$ is the vector of generalized coordinates of the nth element. The numbering $i < j$ is considered to avoid the repetition of the same set of equations.

EQUATIONS OF MOTION

Assuming workless constraints, the equations of motion of the ith element are

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \left[\frac{\partial L}{\partial q}\right] = \left[Q\right] + \sum_{k=1}^{n}\left[\frac{\partial \phi_k^{ij}}{\partial q}\right]\left[\lambda_k^{ij}\right]^t + \sum_{k=1}^{m}\left[\frac{\partial \phi_k^{ji}}{\partial q}\right]\left[\lambda_k^{ji}\right]^t \qquad (14)$$

where, $L = T - V$ is the Lagrangian function,

n,m are the number of constraint equations between elements i and j for $i < j$ and $i > j$, respectively.

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The λ,s are the Lagrange multipliers. There are many λ,s as the number of constraint equations. The equations of motion for the ith element are given in the appendix where they are arranged in a coupled system of non-linear algebric and differential equations of the form

$$f(y^{o}, y) = 0 \tag{15}$$

where y is a vector of N time-dependent variables.

For simultaneous numerical solution of Eq.(15), the total time interval (O,T) is divided into short time intervals $h_i = t_{i+1} - t_i$. The progress from t_i to t_{i+1} is carried out using Gear's algorithm [11] , and for each time step corrector formulas are used such that iterative procedure is continued until all of the Newton differences Δy are below a specified tolerence level.

APPLICATIONS

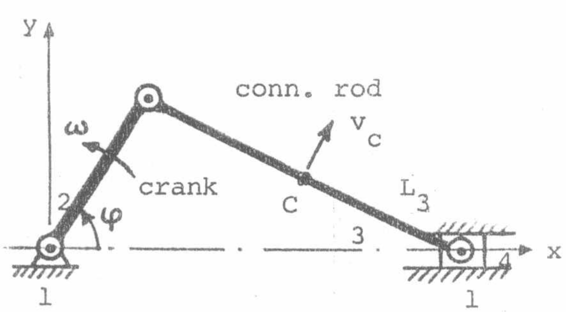

Fig.5. Slider Crank Mechanism

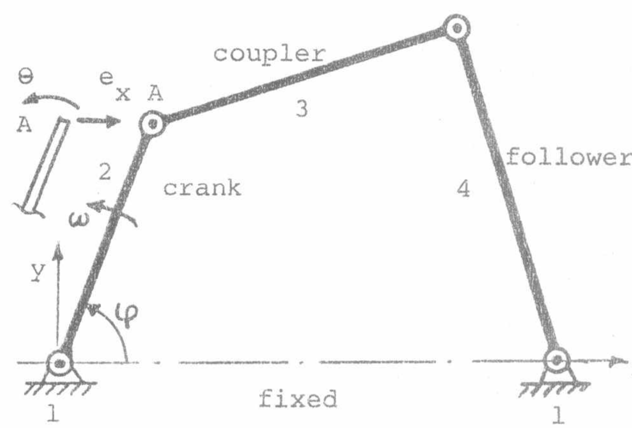

Fig.6. Four Bar Mechanism.

Slider Crank Mechanism (Fig.5)

The characteristics of the chosen mechansim are :

Crank length = 6 in , crank angular velocity = 124.8 rad/sec
connecting rod length=12in,connecting rod diameter= 0.25 in

The two members are made of steel with:

$$E = 30 \times 10^{6} \ \text{Ib/in}^2; \qquad \rho = 0.0007331 \ \text{Ib.sec}^2/\text{in}^4$$

The crank is considered to be rigid and the slider is assumed to be mass-less. In the initial position, $\varphi = 0$, the elastic deformations are assumed to be zero. The elastic deformation v_c at the midpoint of the connecting rod is shown in Fig.7. The results are obtained for two cases of connecting rod modelling. First, it is taken as one element. Secondly, it is divided into two equal elements. The results are compared with those obtained using another method [9] for the same mechanism. One can note the agreement of the results.

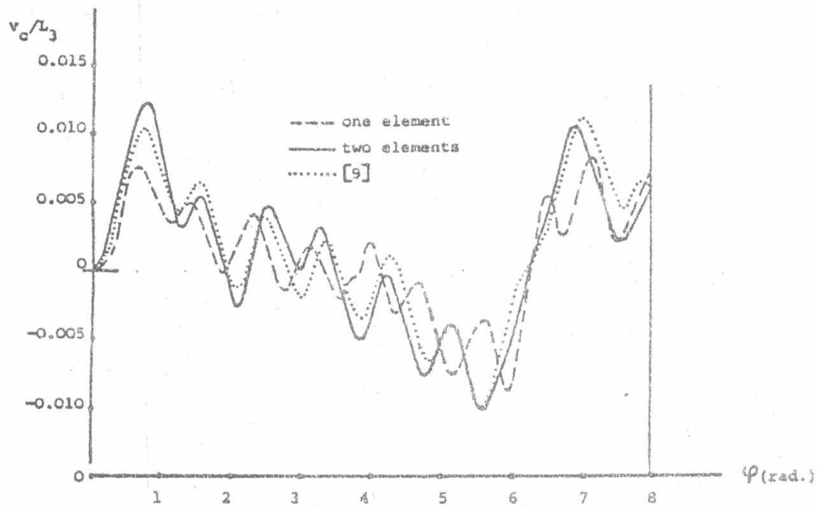

Fig.7. Elastic Deformation at C

Four Bar Mechanism (Fig.6)

All members are considered to be elastic and two equal finite elements for each link are used in the analysis. The characteristics of the mechansim are :

L_1 = 10 in , L_2 = 5 in

L_3 = 11 in , L_4 = 10.5 in

where, L_1, L_2, L_3 and L_4 are the lengths of fixed, crank, coupler and follower links, respectively.

Each member is considered to be a steel rod with

$E = 30 \times 10^6$ Ib/in^2 , $\rho = 0.000725$ Ib.sec^2/in^4

$A = 0.25$ in^2 and $I = 0.0208$ in^4

The input angular velocity is uniform and equals 125 rad/sec. In the initial position, $\varphi = 0$, the elastic deformations are assumed to be zero. The elastic deformations e_x and θ at the joint A are shown in Figures (8) and (9), respectively, where e_x is the x component of deformation at A. The results are in good agrement with those obtained in [5] for the same mechanism.

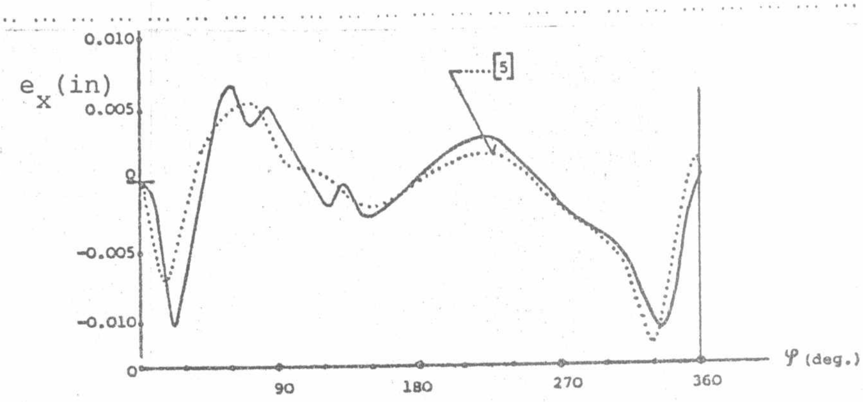

Fig.8. Deformation e_x - Four Bar Mechanism

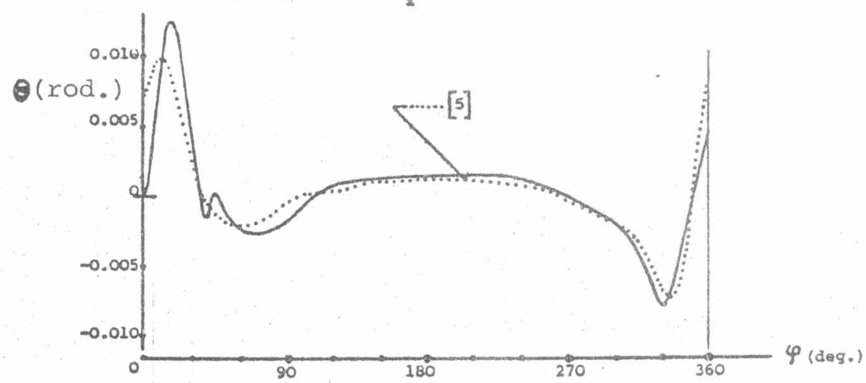

Fig.9. Deformation θ - Four Bar Mechanism.

CONCLUSION

A general formulation is presented for the dynamic analysis of planar mechanisms consisting of elastic members. The compatability conditions of the deformations, at the kinematic pairs, are investigated and introduced into the analysis through Lagrange multipliers. The governing equations of motion combine both the rigid body motions and the small elastic deformations. The results, obtained by applying the suggested technique on numerical examples, show a good agreement with those previously solved using other methods. An important feature of this formulation is that linear combinations of the multipliers give the dynamic reacting forces at the kinematic pairs, which will be the interest of the authors for future work.

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APPENDIX

Denoting
$$A_q = \sum_{k=1}^{n} \frac{\partial \phi_k^{ij}}{\partial q} \lambda_k^{ij} \qquad i < j$$

$$B_q = \sum_{k=1}^{m} \frac{\partial \phi_k^{ji}}{\partial q} \lambda_k^{ji} \qquad i > j$$

the equations of motion of ith element are :

$$* \quad \overset{\circ}{p}_x - Q_x + A_x + B_x = 0$$

$$* \quad \overset{\circ}{p}_y - Q_y + A_y + B_y = 0$$

$$* \quad \overset{\circ}{p}_\theta - \frac{M}{2}\left[\frac{1}{3} d_3(d_1\sin\theta - d_2\cos\theta)(u_1+u_2) - \frac{1}{3}(d_1\cos\theta + d_2\sin\theta)(d_4+d_5)\right] - Q_\theta + A_\theta + B_\theta = 0$$

$$* \quad \overset{\circ}{p}_{u1} + \frac{M}{2}\left[\frac{1}{3}d_3(d_1\cos\theta+d_2\sin\theta) + \frac{1}{5} d_3 d_6 - \frac{1}{70} d_3^{\,2}(u_1+u_2)\right] + \frac{32EI}{L^3}(2u_1-u_2) - Q_{u1} + A_{u1} + B_{u1} = 0$$

$$* \quad \overset{\circ}{p}_{u2} + \frac{M}{2}\left[\frac{1}{3}d_3(d_1\cos\theta+d_2\sin\theta) - \frac{1}{5} d_3 d_6 - \frac{1}{70} d_3^{\,2}(23u_1+33u_2)\right] + \frac{32EI}{L^3}(2u_2-u_1) - Q_{u2} + A_{u2} + B_{u2} = 0$$

$$* \quad \overset{\circ}{p}_{u3} - \frac{M}{2}\left[\frac{1}{5}d_3(d_4-d_5) + \frac{1}{3} d_3^{\,2}(L+6u_3)\right] + \frac{4EA}{L} u_3 - Q_{u3} + A_{u3} + B_{u3} = 0$$

$$** \quad p_x - \frac{M}{2}\left[2d_1 - \frac{1}{3}d_3(u_1+u_2)\cos\theta - \frac{1}{3}(d_4+d_5)\sin\theta\right] = 0$$

$$** \quad p_y - \frac{M}{2}\left[2d_2 - \frac{1}{3}d_3(u_1+u_2)\sin\theta + \frac{1}{3}(d_4+d_5)\cos\theta\right] = 0$$

$$** \quad p_\theta - \frac{M}{2}\left[\frac{1}{6}L^2 d_3 - \frac{1}{3}(d_1\cos\theta+d_2\sin\theta)(u_1+u_2) - \frac{1}{10}(d_4-d_5)(L-2u_3) - \frac{1}{5}d_6(u_1-u_2) + \frac{2}{3}u_3 d_3(L+u_3) + \frac{2}{5}d_3(u_1+u_2)^2 + \frac{1}{14}d_3(u_1-u_2)^2\right] - \frac{MI}{A}\left[d_3 - \frac{1}{L}(d_4-d_5)\right] = 0$$

$$** \quad p_{u1} + \frac{M}{2}\left[\frac{1}{3}(d_1\sin\theta - d_2\cos\theta) + \frac{1}{10}d_3(L-2u_3) - \frac{1}{70}(33d_4+23d_5)\right] + \frac{MI}{2A}\left[\frac{2}{L}d_3 - \frac{1}{15L^2}(94d_4 - 14d_5)\right] = 0$$

$$** \quad p_{u2} + \frac{M}{2}\left[\frac{1}{3}(d_1\sin\theta - d_2\cos\theta) - \frac{1}{10}d_3(L-2u_3) - \frac{1}{70}(23d_4+33d_5)\right] - \frac{MI}{2A}\left[\frac{2}{L}d_3 - \frac{1}{15L^2}(14d_4-94d_5)\right] = 0$$

$$** \quad p_{u3} + \frac{M}{2}\left[\frac{1}{5}d_3(u_1-u_2) - \frac{2}{3}d_6\right] = 0$$

$$*** \quad \dot{x} - d_1 = 0$$

$$*** \quad \dot{y} - d_2 = 0$$

$$*** \quad \dot{\theta} - d_3 = 0$$

$$*** \quad \dot{u}_1 - d_4 = 0$$

$$*** \quad \dot{u}_2 - d_5 = 0$$

$$*** \quad \dot{u}_3 - d_6 = 0$$