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MODAL ANALYSIS BY STATE-SPACE APPROACH IN FREQUENCY DOMAIN

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ABSTRACT

This paper tests a new procedure to identify the eigenvalue problem matrix "A", and thus the system's modal parameters, by utilizing the frequency response of the system in state-space. A sample identification test is performed with added random error in order to simulate real measuring conditions. Results indicate that it is slightly difficult to identify the "A" matrix of an undamped system. The error in that system must be very small for this type of system to be identified. However, when the identification is repeated several times and then averaged, the identification becomes more accurate.

INTRODUCTION

Modal analysis can be defined as the theoretical or experimental analysis of the structure's dynamic characteristics of a mechanical system in terms of its modal parameters; that is, finding the eigenvalues and eigenvectors of the equations of motion which define the mechanical system. The development of modal testing techniques, which took place mainly after World War II , was made principally in the aircraft and space vehicle fields (Budd [1]). Before 1960, frequency response determination for modal analysis was limited to a discrete sine testing method, and mode shape was determined by a simple sine dwell method. Sand patterns and other similar testing techniques were also used to determine nodal (zero amplitude) patterns. During the early 1960's, the most important development was the tracking filter that allowed the use of the mechanical impedance measuring analyzer; mostly known as the transfer function analyzer. The transfer function analyzer measured frequency response by using a sinusoidal excitation signal to excite the mechanical system. This process made it possible to get frequency response plots, and from the poles, natural frequencies, and damping could be estimated. Another improvement was the development of a coquad meter which allowed the real and imaginary components of the frequency response to be measured directly. Instead of using the total amplitude for the mode shape values, the quadrature response could be used to separate closely coupled modes.

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Experimental modal analysis became divided into two fields: (1) the multi-input sine dwell (MISD), and (2) the single input frequency response (SIFRQ).

The multi-input sine dwell test (MISD) was the most popular modal testing method. Several classical papers were published which discuss the calculation of force ratios among shakers, like Kennedy [2], Lewis [3], Asher [4] and Bishop [5] . In this procedure, however only one frequency could be measured at a time, so it was required to have a large number of response transducers in parallel to minimize the test time.

For the single - input frequency response method (SIFRQ) it was necessary to measure a large number of frequency responses. This was a very slow process using existing analog equipment at that time. But, in the late sixties minicomputer systems and the implementation of the fast Fourier transform algorthm in these computers made it possible to develop a Fourier analysis system which could be used to measure frequency response in a small fraction of the time required with the analog equipment. Cooley and Tukey [6] developed the Fast Fourier Transform procedure which made all that possible and did some studies on complex Fourier series machine calculations. Many new excitation and testing methods were developed: impact testing, random, pseudo-random, etc., as a result of these new Fourier techniques.

Experimentally, the present technology is a continuation of the technologies developed during the sixties but much more refined. With the introduction of low cost minicomputers, and the development of software programs for modal testing, for example General Radio Time/Data Division [7] and Hewlett-Packard [8] , more tests are done using digital dequipment. The key to conducting the modal test using a digital signal processing technique is the measurement of the frequency response function between the input forces and the responses at different locations of the structure. Modal parameters for defining the dynamic characteristics of the structure are then extracted from these measured functions. Substantial literature has dealt with this approach, for example, Richardson and Potter [9]; Ramsey [10,11] ; Klosterman and Zimmerman [12] ; Brown and others [13] . Single excitation frequency response methods have probably been developed nearly to their limit Potter [14] . The multi-shaker sine dwell method, as in Craig and Su [15], is also reaching its practical limits. Multi-input methods are currently being developed which are combining the best features of both methods Allemang and others [16]. Also new non-traditional methods have been developed such as Ibrahim time domain (Ibrahim and others [17, 18]), autoregressive moving average (Friedman [19]), direct parameters identification (Maattaneu [20]), and poly reference (Vold and Rocklin [21]).

In the theoretical area, the method presented here and other advances in the area will, hopefully, help in making analysis more accurate, easier to perform and simpler to modify.

THEORETICAL DEVELOPMENT

In the absence of gyroscopic, damping, circulatory forces, the equation of motion of conservative system becomes

$$M\ddot{\bar{q}}(t) + K\bar{q}(t) = \bar{Q} \qquad (1)$$

We can write this equation as [22] :

$$M^*\dot{\bar{x}}(t) + K^*\bar{x}(t) = \bar{U} \qquad (2)$$

Where $\bar{x}(t)$ is the 2n-dimensional state vector:

$$\bar{x}(t) = \left[\dot{\bar{q}}(t)^T \mid \bar{q}(t)^T\right]^T \qquad (3)$$

and

$$M^* = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & I \end{array}\right], \qquad K^* = \left[\begin{array}{c|c} 0 & K \\ \hline -I & 0 \end{array}\right] \qquad (4)$$

2n x 2n are real matrices and where $\bar{U}$ is a forcing vector.

$$U = \left[\begin{array}{c} \bar{Q} \\ \hline \bar{0} \end{array}\right] \qquad (5)$$

The final formulation in state space representation is then given by:

$$\dot{\bar{x}} = A\bar{x} - {M^*}^{-1}\bar{U} \qquad (6)$$

where

$$A = -{M^*}^{-1}K^* = \left[\begin{array}{c|c} 0 & -M^{-1}K \\ \hline I & 0 \end{array}\right] \qquad (7)$$

Assuming that M* is not singular, A is an arbitrary real matrix. Because A is real, if the eigenvalues λ are complex, then they must occur in pairs of complex conjugates and so must the eigenvectors $\bar{x}$.

The procedure presented here is based on the frequency-response technique using the state-space vector method, (Takahashi, Rabins and Auslander [23]). The derivative operator d/dt in equ. (6) is replaced by $j\omega$ to deal with sinusoidal steady-state (or frequency response), where ω (rad/sec) is the angular frequency of the sinusoidal input and $j = \sqrt{-1}$. Finally, the following form for the ith state variable in sinusoidal steady-state is obtained:

$$x_i(\omega) = \alpha_i \sin \omega t + \beta_i \cos \omega t \qquad (8)$$

with

$$\alpha_i = A_i \cos \psi_i, \quad \beta_i = A_i \sin \psi_i \qquad (9)$$

where A_i is the amplitude and ψ_i is the phase of the ith state space variable.

From equation (8), we get:

$$\bar{x}(\omega) = [\bar{\alpha}, \bar{\beta}] \begin{bmatrix} \sin \omega t \\ \cos \omega t \end{bmatrix} \quad (10)$$

where

$$\bar{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n} \end{bmatrix} , \quad \bar{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{2n} \end{bmatrix} \quad (11)$$

The time derivative for $x(\omega)$ is given by:

$$\frac{d}{dt} \bar{x} (\omega) = [\bar{\alpha}, \bar{\beta}] \begin{bmatrix} \omega \cos \omega t \\ \omega \cos \omega t \end{bmatrix} = [-\omega\bar{\beta}, \omega\bar{\alpha}] \begin{bmatrix} \sin \omega t \\ \cos \omega t \end{bmatrix} \quad (12)$$

Finally, the values for $\bar{\omega}$ and $\bar{\beta}$ are found to be (Takahashi, Rabins and Auslander [23]):

$$\bar{\alpha} = (\omega^2 I + A^2)^{-1} (-A\bar{b}_1 + \omega\bar{b}_2) \quad (13)$$

$$\bar{\beta} = (\omega^2 I + A^2)^{-1} (-A\bar{b}_2 - \omega\bar{b}_1) \quad (14)$$

The amplitude is given by:

$$|x_i| = \sqrt{\alpha_i^2 + \beta_i^2} , \; i = 1, 2, \dots 2n \quad (15)$$

The phase angle is obtained from:

$$\phi = \tan^{-1} \frac{\alpha_i}{\beta_i} \quad (16)$$

In frequency domain and for non-homogeneous equations, we start the procedure of identifying the eigenvalue problem matrix with equation (7) to (14). That process would describe reationships for sinusoidal steady-state.

Equations (13) and (14) were derived by generalizing a vector input, assuming:

$$\bar{u}(\omega) = [\bar{b}_1, \bar{b}_2] \begin{bmatrix} \sin \omega t \\ \cos \omega t \end{bmatrix} \quad (17)$$

where $\bar{b}_1$ and $\bar{b}_2$ are prescribed magnitudes with each as a 2n dimensional vector.

Substituting equations (10), (12) and (17) into equation (7), we find after some manipulation that:

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$$-\omega\bar{\beta} = A\bar{\alpha} + \bar{b}_1 \tag{18}$$

$$\omega\bar{\alpha} = A\bar{\beta} + \bar{b}_2 \tag{19}$$

Metwalli [24] shows that by measuring at 2n the ω_i, we can get $\bar{\alpha}$, and $\bar{\beta}$ for any ω_i, then arranging the different vectors in equation (19) to obtain:

$$A = [\omega_1\bar{\alpha}_1 - \bar{b}_{2,1} \mid \omega_2\bar{\alpha}_2 - \bar{b}_{2,2} \mid \cdots \mid \omega_{2n}\bar{\alpha}_{2n} - \bar{b}_{2,2n}]\ [\bar{\beta}_1\ \bar{\beta}_2 \cdots\ \bar{\beta}_{2n}]^{-1} \tag{20}$$

Also from equation (18), one can get [24]

$$A = [\omega_1\bar{\beta}_1 - \bar{b}_{1,1} \mid \omega_2\bar{\beta}_2 - \bar{b}_{1,2} \mid \cdots \mid \omega_{2n}\bar{\beta}_{2n} - \bar{b}_{1,2n}]\ [\bar{\alpha}_1\ \bar{\alpha}_2 \cdots \bar{\alpha}_{2n}]^{-1} \tag{21}$$

The following step is to add a random error to the response so that to simulate real measuring conditions. Each of the ith columns of the first RH matrix in equation (19) has been changed to:

$$\text{ith column} = (\omega_i\bar{\alpha}_i - \bar{b}_{2,i} - \bar{b}_{2,i})\ [1 + \%\text{R.E.}] \tag{22}$$

where R.E. is the percentage of random error added. The random function used in the computer program causes an internal pseudorandom number generator to output a random number from 0 to 1. The probability distribution of these random numbers is uniform, which has been used as such for simplicity. Modal parameters are then obtained as the eigenvalues and eigenvectors of A.

SAMPLE TEST

A number of Cam follower systems that have been used for automotive and aircraft piston engines have a complex follower system involving two levers for motion amplification. In this example, we will consider a double lever Cam mechanism with a four degree of freedom dynamic system model. A study about modeling this system has been done by Teser and Matthew [25] and the equations of motion and detailed modeling procedures are also explained by Barkan [26] . The details of the double lever Cam follower system and system model are shown in Figure 1. The differential equations of motion of this system are written in terms of displacement variables q_i, lumped masses m_i, and linear spring stiffness K_i. Damping coefficients are neglected by Tesar and Matthew [25] for simplicity.

In a matrix form the equation of motion is

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} k_1 y \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

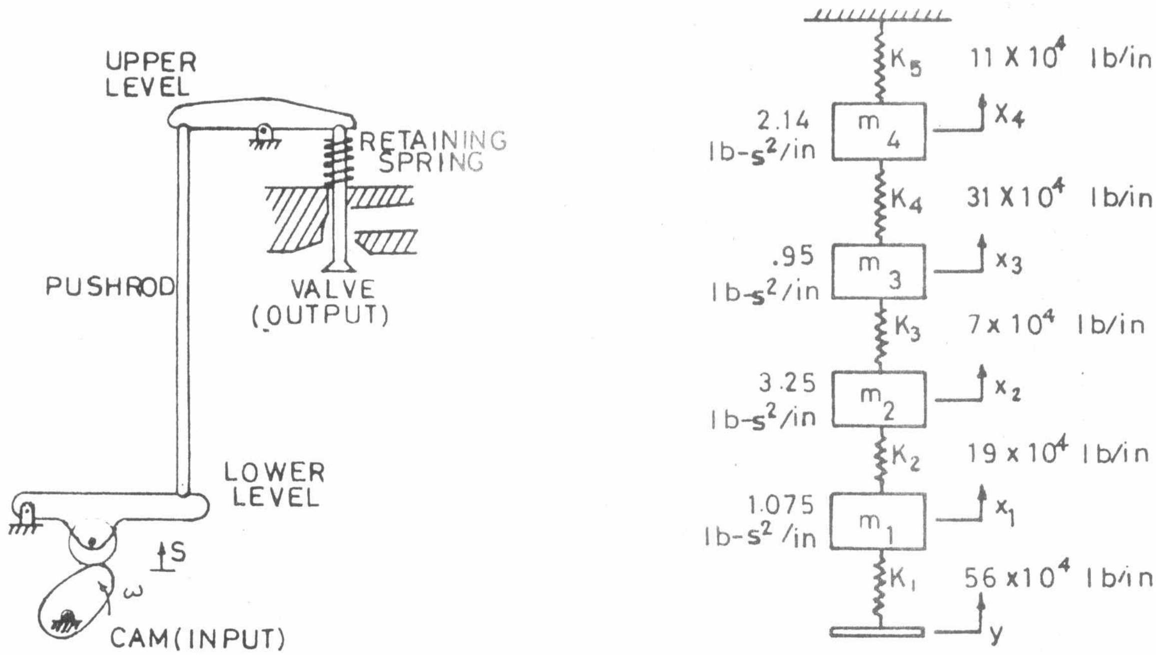

Fig.1: Double lever cam follower system and model.

Next, we consider the data of a sample calculation given by Young and Shoup [27] and shown in Fig. 1 .

In state space representation, the system matrix (equation 7 and using computer inversion) is given by

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & -697,674 & 176,744 & 0 & 0 \\ 0 & 0 & 0 & 0 & 58,461 & -80,000 & 21,538 & 0 \\ 0 & 0 & 0 & 0 & 0 & 73,684 & -400,000 & 326,315 \\ 0 & 0 & 0 & 0 & 0 & 0 & 144,859 & -192,261 \\ 1 & -1.776\times10^{-15} & 2.664\times10^{-15} & 1.776\times10^{-15} & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.065\times10^{-14} & -7.105\times10^{-15} & 0 & 0 & 0 & 0 \\ -3.552\times10^{-15} & 7.105\times10^{-15} & 1 & 0 & 0 & 0 & 0 & 0 \\ -3.552\times10^{-15} & 5.329\times10^{-15} & -2.842\times10^{-14} & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (24)$$

A computer program for state space frequency response which follows eqs. (13) and (14) was used. The input data was the matrix A (equation 24) and the input is defined by:

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{aligned} \omega_{increment} &= 10 \text{ cycles/sec} \\ \omega_{maximum} &= 1{,}000 \text{ cycles/sec.} \end{aligned}$$

We used a unit sinusoidal and cosinusoidal input and a range of frequencies that includes the values of the eigenvalues. This is because if we input only sine or cosine, we will have a singular matrix during the manipulation process.

A computer program for system identification used equations (20) and (21) to get the original matrix A of this example. A 0.5 (10^{-4}) or 0.005% random error was added to the frequency response solution, but the original matrix was not obtained. Finally, different percentages of random error were added to the solution. Only in the last case, 0.5 (10^{-10}), the identified matrix is exactly the same as the original. For the other cases between 0.5(10^{-4}) and 0.5 (10^{-8}), a graph was plotted to observe the behaviour of both the ral and complex part of the eigenvalues. This is shown in Fig. 2.

The identification computer program was then used 100 times and averaged for the added 0.5 (10^{-4}) random error. The matrix obtained was very similar to the original matrix A.

In addition to that, the error matrix was calculated for every case, and the eigenvalues and eigenvectors are obtained. It can be noticed that the complex part of the eigenvalues of the error matrix is very small compared with the eigenvalues of the original matrix A. The eigenvalues of the real part are larger compared with the original eigenvalues, and for a 0.5 (10^{-4}) random error there is one eigenvalue which is large compared with the values of the original matrix A.

The simulated system responses for other damped non-gyroscopic systems show that the solution is stable [28]. It was seen that the peaks occur for frequencies corresponding to the values of imaginary parts of the complex conjugate roots, in frequency versus ampltude graphs of original systems. The error sigenvalues obtained for the added random errors (0.2% to 1.0%) are not bigger than 5% of the original eigenvalues. The identification of the system matrix was performed one hundred times and averaged with a 0.5% added random error for all cases. The matrices obtained through the new identification process [28] are very good approximations to the original system matrix. The input used for the identification process was a unit sinusoidal force. The process could also be done with a unit cosinusoidal force. Both inputs can be used at the same time, with same results obtained.

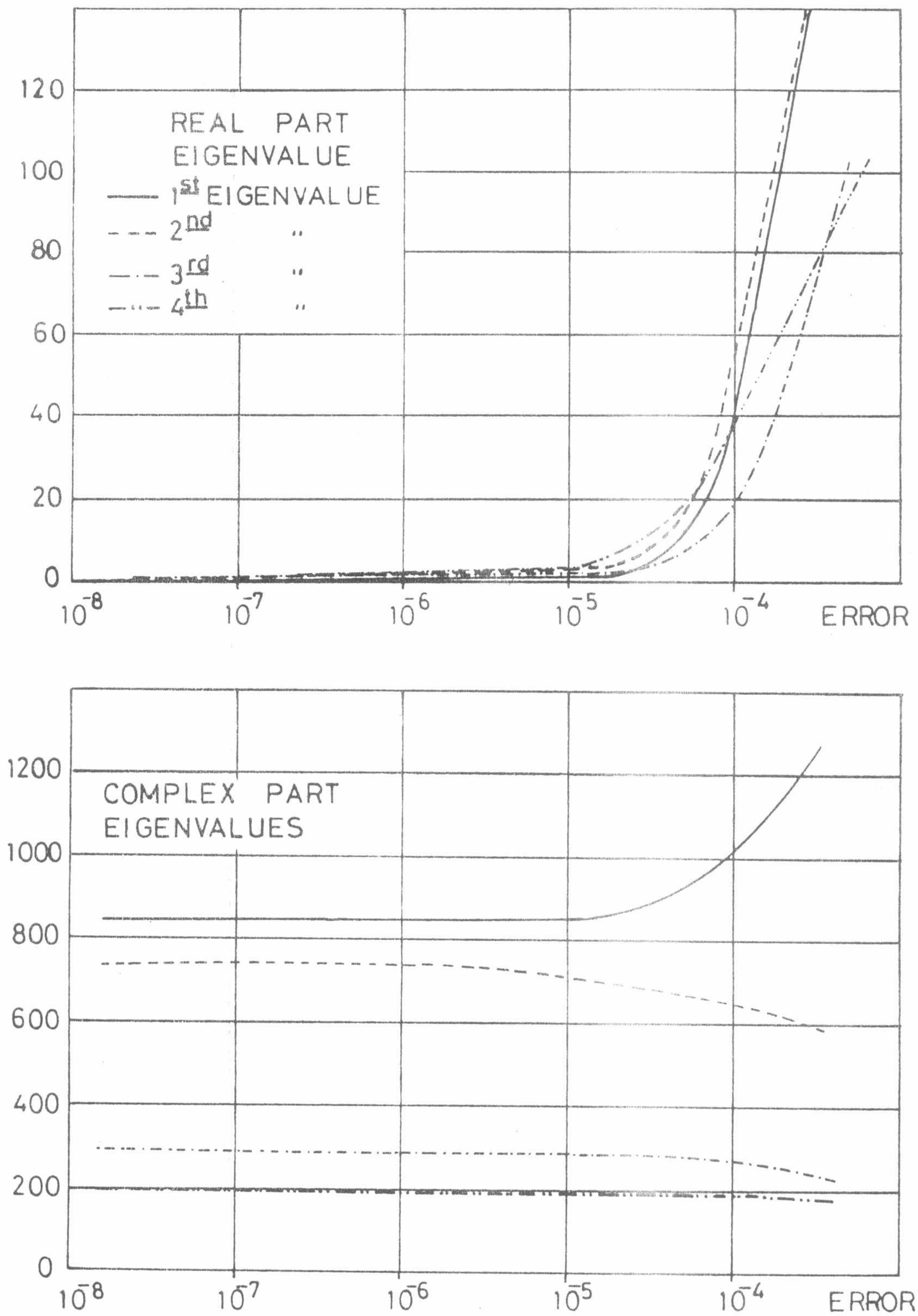

Fig. 2 : Matrix error eigenvalues versus random error

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CONCLUSION

In this paper we have presented a method to determine the modal parameters of a system. This method was verified by finding the response of a sample case and the identification tests were then performed.

A conservative non-gyroscopic system was modeled and solved. The identification of the system matrix was then obtained. With unit sinusoidal or cosinusoidal force as an input, we will find singularities during the manipulation process that will not permit us to get a solution. This is caused because of the lack of damping in this particular case. The error added in order to simulate real measuring conditions had to be much smaller than in damped cases in order to get the original matrix A. Therefore, measurements for such systems should be very accurate so as to use such technique effectively for undamped systems. For realistic measurement errors, results indicate that averaging can improve the accuracy of the identification process.

For cases of damped non-gyroscopic systems, however, the results obtained for identification were accurate.

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NOMENCLATURE

A	System's matrix in state space
b_1	Forcing input (sin ωt)
b_2	Forcing input (cos ωt)
I	Identity Matrix
i	Interger number
K*	Stiffness matrix in state space
K	Stiffness matrix
M*	Mass matrix in state space
M	Mass matrix
n	Dimension of a vector matrix
$\bar{Q}$	Generalized force vector
$\bar{q}$	Generalized coordinate vector
t	time
$\bar{U}$	Input vector
$\bar{u}$	r-dimensional input vector
$\bar{x}$	State Vector
α	Real part of complex eigenvalue
β	Imaginary part of complex eigenvalue
λ	Eigenvalue of the system's matrix
φ	Phase angle
ψ	Phase of the state variable
ω	Frequency