6

AN ALGORITHM FOR THE SOLUTION OF AVISCO PLASTIC MODEL IN A STEADY STATE OF CREEP USING A NON CONFORM FINITE ELEMENT

By

Mohamed Rafik ABDEL FATAH *

Abstract

A non-conform finite element proposed for viscous fluid flow satisfying zero divergent condition is adapted to our model. This needs some numerical aspects aspects which characterize our algorithm and precise the method of solution on digital computers.
The variational formulation of the model permits the use of optimization techniques. A descent technique, thqt of Flecher-Reeves with re-initalization and a proposed line search t-chnique are used. For our mo[illegible] the vanishing divergence condition is not sufficient to sustain our problem as an unconstrained optimization problem. A penalty method (exterior penalty function) is now necessary for the numerical solution.
Apractical example is solved using our algorithm and the corresponding stress distributions are obtained.

Introduction:

Consider a two dimension problem consisting of a domain $\Omega \subset R^2$ occupied by the continuum medium. A suface traction T(t) is applied on a part Γ_1 of Γ. (T function of time t). On the complementary part Γ_2 of Γ a strictly positive measure, a field of diplacement rates $\hat{u}(t)$ is applied. The variational formulation of the problem made in |1| will be considered here, in which the problem reduces to

$$\min_{(\sigma,s)\,\varepsilon I'^{o} \times S} G(\sigma_o + \sigma, s) = \min \int_\Omega G_1\,(\sigma_o + \sigma_1\; s)\; dx \qquad (P)$$

where σ_o is the stress feild satisfying the equilbruim conditions with the exterior load.

The product space σ_x S= $L_{\tilde{G}}$ $((D'(\Omega))^4, (D'(\Omega))^m)$ is a Banach space spanned by the domain of $\tilde{G}$ where (see [1] , III)

$$\tilde{G}(\sigma,s) = Co\,(\min \int_\Omega (G_1(x,\sigma_o+\sigma,s)\,dx, \int_\Omega G_1(x,\sigma_o-\sigma,-s)\,dx\,))$$

$$I'^{o} = \{\sigma\varepsilon\sigma : \sigma_{ij}\;,\; D^*\sigma = 0 \text{ in } \Omega,\; \sigma_{ij}\, n_j = 0 \text{ on } \Gamma_1 \} \qquad (1)$$

Dr.Eng.,Department of Math.,M.T.C. , Cairo.

which is a subspace of σ

where $D^{*} = -(\partial_1 \sigma_{11} + \partial_2 \sigma_{12}, \partial_1 \sigma_{21} + \partial_2 \sigma_{12})$ (2)

The numerical solution of (p) using of finite elements of faced with an essential difficulty that of the null divergence conditions of (2). This point can be achieved in two ways

(i) If we use a suitable lagangian formulation the problem reduces to the search for a saddle point. The most convenient finite element in the scase will be that of mixed and hybride type (cf. [6], [12]).

(ii) Use finite elements of null divergence (introduced in [5] for fluid flow). The problem will then reduces to an unconstiained minimization problem this technique will be used throughout this work

2. Finite Element Approximate Problem

Ω an open bounded polygon in R^2, (τ_h) be a regular family of triangulation on , i.e. a converage family $\tau_h = \{K_i\}$ $i \in I_h$, $h > o$, by closed triangles K_i satisfying:

(i) $K_i \wedge K_j$, $i \neq j$ is either empty or a side K_{li} or a vertex a_{li}

(ii) The maximum diameter of elements of τ_h is less than or equal h.

(iii) There exist $\nu > o$ such that for all $h > o$ and all $k \in \tau_h$

$$\frac{\text{diam }(K)}{\rho_k} \leq \nu$$

where ρ_k is the diameter of the circumscribed circle.

(iv) The triangulation τ_h is compatible with the decomposition Γ_1, Γ_2 of the boundary Γ of Ω, i.e. $\Gamma_i \wedge \partial K (i=1,2)$ is either empty or a union of sides of some K.

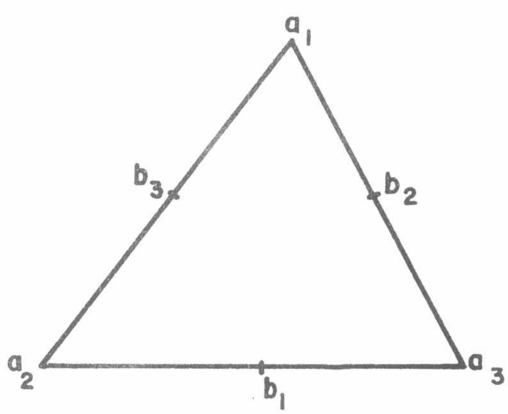
Let P_1 (A) be the space of two variable polynomials of degree less than or equal to one defined over A R^2. P_1 is therefore of dimension three. let also P_1^h be the space of functions defined on Ω such that

$$P_1^h = \{v \in R^{\Omega} : v|_K \in P_1(K),\ K \in \tau_h\} \quad (2)$$

We denote by b_i ($i=1,2,3$) meridians of the triangle K opposing the vertex a_i.

We can easily verify that the set $\Sigma_K = \{b_i\}$ $1 \leq i \leq 3$ (Cf. [4]) is P_1 - unisolvant (Cf. [4]) while $\Sigma_h = \bigcup_{K \in \tau_h} \Sigma_K$ is the set of nodes (of number N_h) of τ_h.

We denote the finite element space adapted to our problem by X_h°



F ig.1.

$$X_h^o = \{ v_h \varepsilon\, P_1^h : v_h \text{ continuous at } b_i,\ b_i \varepsilon \Sigma_h,\ v_h(b_j)=o \ \ b_j \varepsilon \Sigma_h\ \Gamma \} \quad (4)$$

Let $r_h : \sigma \rightarrow (P_1^h)^4$ be a linear mapping defined by

$$\int_{k'} (r_h v)_j \, d\Gamma = \int_{k'} v_j \, d\Gamma \qquad j=1,2,3,4$$

for all sides K' of the triangles $K \varepsilon \tau_h$. Then we have for every $v \varepsilon C^1$ (.)

$$\int_k \operatorname{div} ((r_h v)_i, (r_h v)_{i+1}) dx = \int_k \operatorname{div} (v_i, v_{i+1}) dx \quad i=1,2,3 \qquad (5)$$

We denote $D\ v = -(\operatorname{div}(v_1,v_2), \operatorname{div}(v_3,v_4))$, and $\int_k D\,(r_h v)\, dx = \int_k d\ v\ dx$ (6)

A direct consequence of (4),(5) and (6) is the following lemma which assure the convergence of the numerical solution (patch test).

Lemma:

For all common side K' of any two adjacent triangle K_i, K_j we have:

$$\int_{k'} (v_{h,i} - v_{h,j})\, d\Gamma = 0 \qquad \forall\, v_h \varepsilon\, X_h^o \qquad (7)$$

where $v_{h,l}$ l=1,2, is the trace of the restriction of v_h to K_l. Further more, if $K' \varepsilon \Gamma$, we have

$$\int_{k'} v_{h,i}\, d\Gamma = 0 \qquad \forall v_h \varepsilon\, X_h^o \qquad (8)$$

We now in a position to construct the needed finite element space

$$Z_h^o = \{ v_h \varepsilon\, (X_h^o)^4 : \int_k D^* v_h \, dx = 0, \quad K \varepsilon \tau_h \} \qquad (9)$$

which can be reduced to that introduced by. Crouzeix Raviart [5] This space is characterized by the conditions of null divergence and trace in the following manner:

Let A,B be the linear mappings

$$\sigma = [\sigma_{ij}] \overset{A}{\mapsto} ((A\sigma)_1, (A\ \sigma_z)_2) = ([\sigma_{i1}], [\sigma_{i2}]) \qquad (10)$$

$$(\vec{p}, \vec{q}) \overset{B}{\mapsto} (\operatorname{div} \vec{p}, \operatorname{div} \vec{q}) \qquad (11)$$

consequently

$$-D^*\sigma = (B \circ A)\sigma \qquad (12)$$

So the introduced finite element (in our work) is in fact a product of the Crouziex-Raviart finite element of σ and $A\sigma$. The functional spaces adjoint to our problem takes the form:

Given D_h^* be the mapping defined by:

$(D^*_h v)|_k = D^*(v|_k) \qquad \forall v \in (p_1^h)^4$

If $v \in (X_h^o)^4$ then $D^* v$ is τ_h - Stepped function.

Denote by $\tilde{\sigma}_h$ the stress field statisfies the equilbruim condition (w.r.t) exterior forces) as follows:

$$\text{Let } Y_h = \{ v \in (p_1^h)^2 : v(b_i) = 0 , \quad b_i \in \Sigma_h \wedge \Gamma_1 \} \tag{13}$$

then for every $\dot{u} \in Y_h$, we have

$$\sum_{k \in \tau_h} \int_K (D^* \tilde{\sigma}_h|_k / \dot{u}) dx + \int_k (p/\dot{u})_2 \, dx + \int_{\Gamma_1 \wedge k} ((\tilde{\sigma}_h . n + T)/\dot{u})_2 \, d\Gamma = 0 \tag{14}$$

Apply now the stokes formula on each finite element taking into considera-tion that the elements satisfy fthe patch test, we get

$$\sum_{k \in \tau_h} \int_k (\tilde{\sigma}_h / D^*(\dot{u}|_k))_4 \, dx + \int_k (p/\dot{u}|_k)_2 \, dx + \int_{\Gamma_1 \wedge k} (T/\dot{u}|_k)_2 \, d = 0 \tag{15}$$

We can now take for $\tilde{\sigma}$ one of the elements of the affine variaty

$$\{\sigma_h \in (p_1^h) : D^*_h \sigma_h + p|_{Y_h} = 0 \text{ and } \sigma_h . n + T|_{\text{Trace } Y_h} = 0 \}$$

Finally we define the finite dimensional space S_h as a subspace of S by :

$$S_h = (X_h^o)^m \quad (\text{m: No of internal parameters}) \tag{16}$$

The problem (P) is now approximated as follows:
determine $(\underline{\sigma}_h, \underline{S}_h)$ that

$$\text{minimize } G(\tilde{\sigma}_h + \sigma_h, S_h) \text{ on } Z_h^o \times S_h \qquad (P_h)$$

Let $\{p_i\}_{1 \leq i \leq 3}$ be the canonical basis of $p_1(k)$, $k \in \tau_h$ with

$$p_i(x_1, x_2) = 1 - 2 \lambda_i(x_1, x_2) \qquad 1 \leq i \leq 3 \tag{17}$$

where λ_i are the barycenteric coordinates w.r.t. the vertex a_i
One can easily verigly that

$$\int_k \lambda_i(x) \, dx = \frac{1}{12} (1 + \delta_{ij}) \Delta K \tag{18}$$

$$\int_k p_i(x) \, dx = \frac{1}{3} \delta_{ij} \Delta_k , \Delta_k = \text{area of } K \tag{19}$$

The later results shows that the functions P_i are orhtogonal in $L^2(K)$.
So one can define on each $K_j \in \tau_h$ the functions Fig.2.

$$W_i^{1,j} = \frac{1}{|a_i a_{i+1}|} P_{i+2} \vec{n}_{i+2} + \frac{1}{|a_i a_{i+2}|} P_{i+1} \vec{n}_{i+1}$$

(relative to the node a_i)

$$W_i^{2,j} = \frac{1}{|a_{i+1} a_{i+2}|} P_i \overrightarrow{a_{i+1} a_{i+2}} \tag{21}$$

(relative to the meridians b_i)

where $\vec{n}_i$ is the normal to the side $\overline{a_{i+1}\,a_{i+2}}$ (modulo 3)

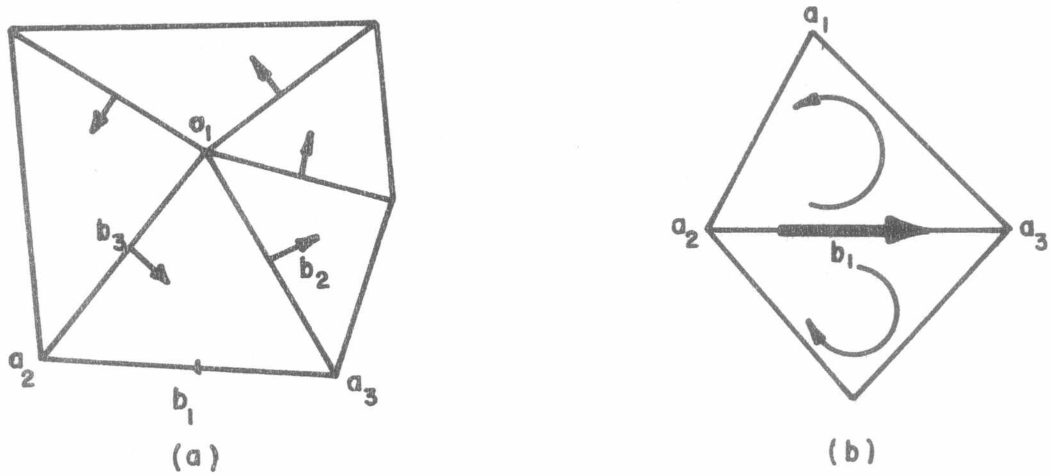


Fig.2.

Using Gauss Formula we get

$$\Delta_j \ \mathrm{div}\ (w_i^{1,j}) = \int_{K_j} \mathrm{div}\ (W_i^{1,j})\ dx = \int_{K_j} W_i^{1,j}\cdot\vec{n}\ \ d\Gamma = 0 \quad (i=1,2,3)$$

So div ($W_i^{1,j}$) = 0 over K_j $1 \leq i \leq 3$

Similarly div ($W_2^{2,j}$) = 0 over K_j $1 \leq i \leq 3$

Numerical Aspects and Considerations

1. One can easily verify that the functions $W_i^{l,j}$ (l=1,2) are dependent, so it is possible to eliminate one of the unknowns $W_i^{l,j}$ $1 \leq i \leq 3$
   Practically, this elimination is done at each node common to a set of triangles which partitioning Ω as in Fig.3. Consequently, for each K_j a set of basis in $(X_h^o)^2|_{kj}$ is defined and the set $\{W_i^{e,j}\}$ is now reduced to only five elements.

At the computational level this reduction of the number of unknowns is of great importance when Card τ_h is very large.

2. Another reduction throughout the problem is attained by taking S_h as a space of functions τ_h-stepped (i.e of class P_o on K). This is acceptable since the elements of S must satisfy only some integrability conditions.

3. The symmetry property of ($\sigma_{ij} = \sigma_{ji}$) is not satisfied by the finite element functions ($W_i^{k,j}$, $W_n^{r,i}$). This difficulty may be over comed at the level of numerical solution using a suitable penalty method.

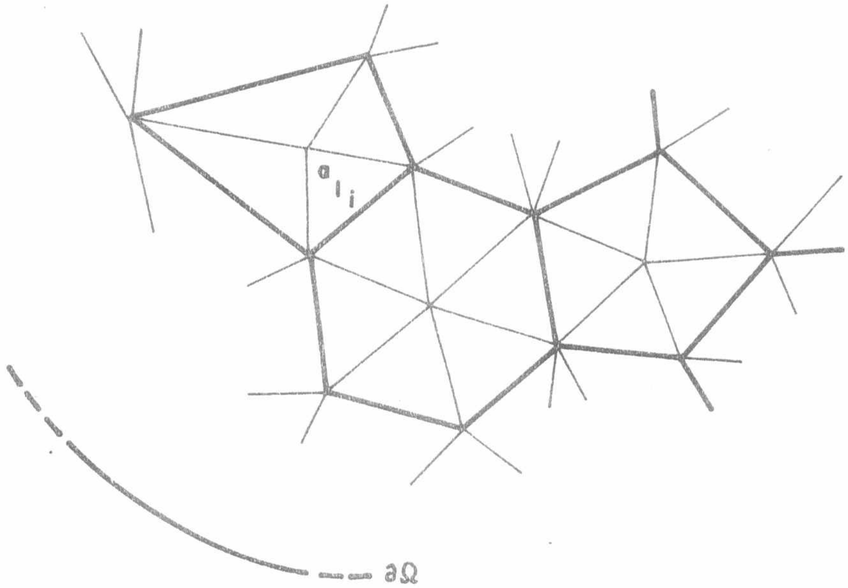


Fig.3.

4. The boundary conditions are treated as follows:

   (i) Determination of $\tilde{\sigma}_h$(densoted here as σ) satisfying (div($\tilde{\sigma}_{11},\tilde{\sigma}_{12}$), div ($\tilde{\sigma}_{21},\tilde{\sigma}_{22}$))= 0 (for null body forces [3])is realized at the computational level by constructing scalar functions Z_1, Z_2 such that:

$$\mathrm{rot}\ (Z_1\ \vec{k}) = (\tilde{\sigma}_{11},\tilde{\sigma}_{12})\ ,\quad \mathrm{rot}\ (Z_2\ \vec{K}) = (\tilde{\sigma}_{21},\tilde{\sigma}_{22})$$

Where $\vec{K}$ is the unit normal to surface Ω. The vector functions $\tilde{p} = (\tilde{\sigma}_{11},\tilde{\sigma}_{12})$, $\tilde{q} = (\sigma_{21},\sigma_{22})$ can be taken as elements of $(P_1^h)^2$

   (ii) Zero boundary conditions do not imply more elimination of the unknowns defined on $K\wedge\Gamma$. To show this, let $\vec{q}_h = (q_{h1}, q_{h2})$ be an approximation of $\vec{q}$ in Z_h^o, then we have

$$\vec{q}_h = \sum_{i=1}^{3} \alpha_i W_i^1 + \sum_{i=1}^{3} \beta_i W_i^2 \qquad (22)$$

and

$$\vec{q}_h|_\Gamma = \alpha_1 W_1^1|_\Gamma + \alpha_2\ W_2^1|_\Gamma + \beta_3\ W_3^2|_\Gamma$$

$$= \left(\frac{\alpha_1 - \alpha_2}{\ell_3}\right) \vec{n}_3 + \frac{\beta_3}{\ell_3} \vec{K}_3$$

where l_i is the length of the side apposing the vertex a_i, Fig.1. hence $q_h|_\Gamma = 0$ implies that only $\beta_3 = 0$, where as $\alpha_1 = \alpha_2$.

5. The number of unknowns define the stress field is $2(N_a + N_b - N_e) + 1$, where N_a is the number of interior vertices of the triangles $K \in T_h$, N_b is the number of interior nodes and N_e is the number of eliminated nodes according to the above rule .

6. The number of unknowns to each component of type P_1^h of an internal perameter is $3N_b$. This number may be reduced to N_b taking $S_h = P_o^h$ (piecewise continuous functions).

6

(relative to the meridians b_i)

where $\vec{n}_i$ is the normal to the side $\overline{a_{i+1}\, a_{i+2}}$ (modulo 3)

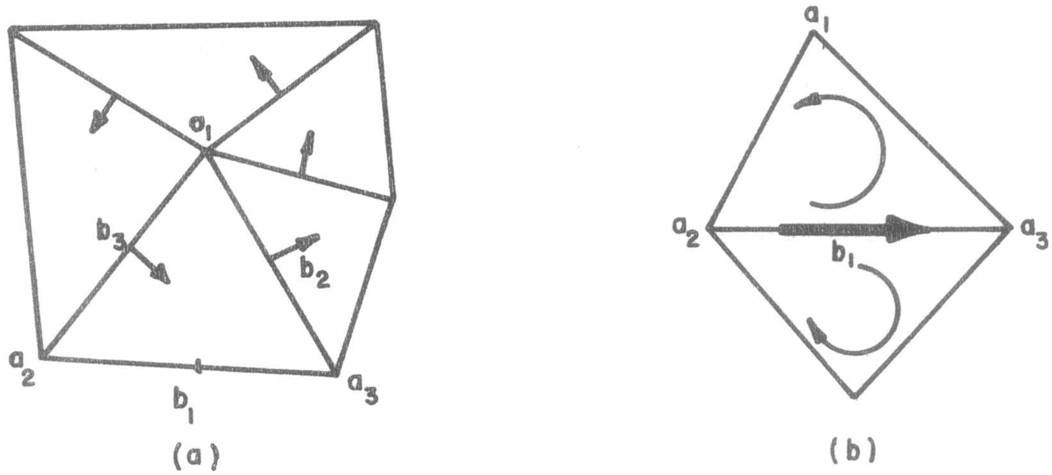


Fig.2.

Using Gauss Formula we get

$$\Delta_j \operatorname{div} (w_i^{1,j}) = \int_{K_j} \operatorname{div} (W_i^{1,j})\, dx = \int_{\partial K_j} W_i^{1,j} \cdot \vec{n}\; d\Gamma = 0 \quad (i=1,2,3)$$

So div $(W_i^{1,j}) = 0$ over K_j $1 \leq i \leq 3$

Similarly div $(W_2^{2,j}) = 0$ over K_j $1 \leq i \leq 3$

Numerical Aspects and Considerations

1. One can easly verify that the functions $W_i^{l,j}$ $(l=1,2)$ are dependent, so it is possible to eliminate one of the unknowns $W_i^{1,j}$ $1 \leq i \leq 3$
   Practically, this elimination is done at each node common to a set of triangles which partitioning Ω as in Fig.3. Consequently, for each K_j a set of basis in $(X^o_h)^2|_{kj}$ is defined and the set $\{W_i^{e,j}\}$ is now reduced to only five elements.

At the computational level this reduction of the number of unknowns is of great importance when Card τ_h is very large.

2. Another reduction throughout the problem is attained by taking S_h as a space of functions τ_h-stepped (i.e of class P_o on K). This is acceptable since the elements of S must satisfy only some integrability conditions.

3. The symmetry property of $(\sigma_{ij} = \sigma_{ji})$ is not satisfied by the finite element functions $(W_i^{k,j}, W_n^{r,i})$. This difficulty may be over comed at the level of numerical solution using a suitable penalty method.

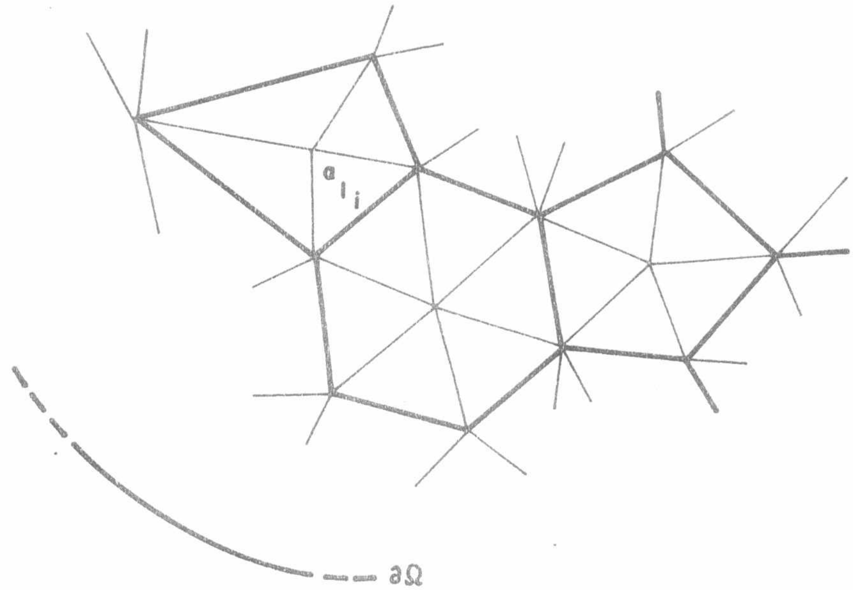


Fig.3.

4. The boundary conditions are treated as follows:

(i) Determination of $\tilde{\sigma}_h$(densoted here as σ) satisfying $(\mathrm{div}(\tilde{\sigma}_{11},\tilde{\sigma}_{12}),\ \mathrm{div}\ (\tilde{\sigma}_{21},\tilde{\sigma}_{22}))= 0$ (for null body forces [3])is realized at the computational level by constructing scalar functions Z_1, Z_2 such that:

$$\mathrm{rot}\ (Z_1\ \vec{k}) = (\tilde{\sigma}_{11},\tilde{\sigma}_{12})\ ,\ \mathrm{rot}\ (Z_2\ \vec{k}) = (\tilde{\sigma}_{21},\tilde{\sigma}_{22})$$

Where $\vec{K}$ is the unit normal to surface Ω. The vector functions $\tilde{p} = (\tilde{\sigma}_{11},\tilde{\sigma}_{12})$, $\tilde{q} = (\sigma_{21},\sigma_{22})$ can be taken as elements of $(P_1^h)^2$

(ii) Zero boundary conditions do not imply more elimination of the unknowns defined on K∧Γ. To show this, let $\vec{q}_h = (q_{h1}, q_{h2})$ be an approximation of $\vec{q}$ in Z_h^o, then we have

$$\vec{q}_h = \sum_{i=1}^{3} \alpha_i W_i^1 + \sum_{i=1}^{3} \beta_i W_i^2 \qquad (22)$$

and

$$\vec{q}_h|_\Gamma = \alpha_1 W_1^1|_\Gamma + \alpha_2\ W_2^1|_\Gamma + \beta_3\ W_3^2|_\Gamma$$

$$= \left(\frac{\alpha_1 - \alpha_2}{\ell_3}\right)\ \vec{n}_3 + \frac{\beta_3}{\ell_3}\ \vec{K}_3$$

where l_i is the length of the side apposing the vertex a_i, Fig.1. hence $q_h|_\Gamma = 0$ implies that only $\beta_3 = 0$, where as $\alpha_1 = \alpha_2$.

5. The number of unknowns define the stress field is $2(N_a + N_b - N_e) + 1$, where N_a is the number of interior vertices of the triangles $K \in T_h$, N_b is the number of interior nodes and N_e is the number of eliminated nodes according to the above rule .

6. The number of unknowns to each component of type P_1^h of an internal perameter is $3N_b$. This number may be reduced to N_b taking $S_h = P_o^h$ (piecewise continuous functions).

6

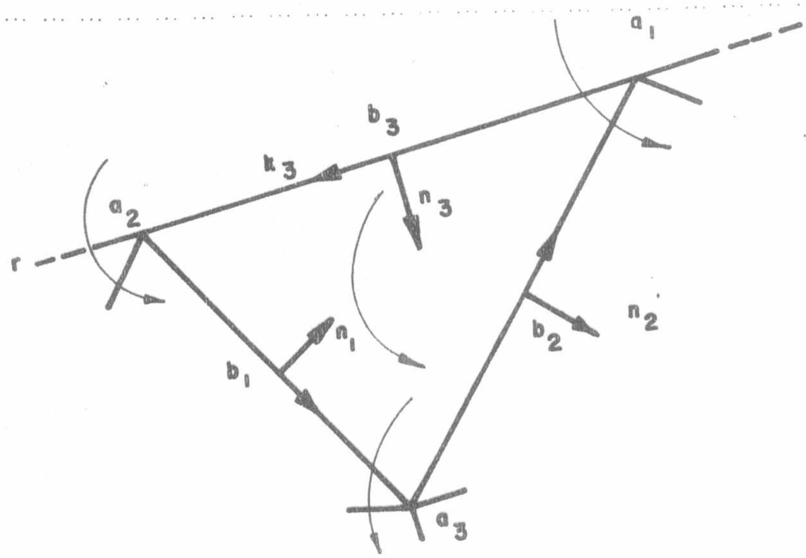


Fig. 4

7. Our algorithm may be constructed as a descent algorithm taking into consideration the results mentioned above together with a line search technique proposed by the author [2]. For the example solved here Fig.4. (in which $G_1(\sigma\ S) = \sum_{i,j}^{2} (\sigma_{ij} - S_{ij})^2$, 1=1) a Fletcher-Reeves algorithm with reinitialization, also an exterior penalty method is used.

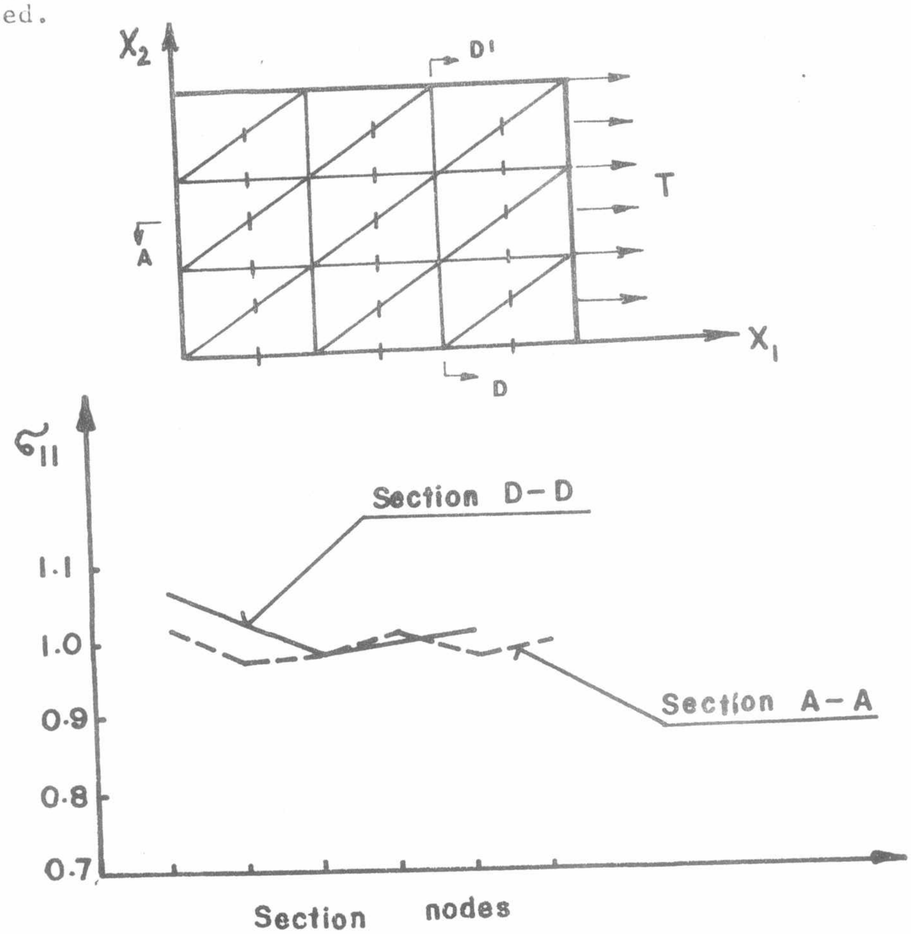


Fig.5.

References:

1 ABD EL FATTAH, M. Rafik. "Sur un Model de Viscoplasticité, Existence et approximation de l'état de Fluage stationnaire " Thèse, Toulouse, 1980.

2. ABDEL FATTAH, M. Rafik." Aline search technique for descent algorithms" Int.J. of computer mathematics, Vol, 12, No 1, 1982.

3. H.A. BRUKE & W.D. NIX," A numerical analysis of void grouth in tension creep" Int. J. of solids structures, Vol. 15, No.1 (1979).

4 P.G. CIARIET "The finite element method for elliptic proble,s" North-Holland (1978)

5 M.CROUZEIX and P.A. RAVIART "Conforming and non conforming finite element methods for solving the stationary stokes equations "R.A.I.R.O. (1973).

6 V.GIRAULT & P.A. RAVIART "Finite element approximation of the Navier-Stokes Equations" Springer - Verlag (1979).

7 B.NAYROLLES."Quelques applications Variationnels de la theorie des functions duales a la mecanique des solides", J. de mécanique, vol. 10, No. 2, (1971).

8 A.R.S. PONTER "Convexity and associated continuum properties of a class of censtitutive relationships", J.de mécanique, Vol. 15, No 4 (1976).

9 A.R.S. PONTER, "General theorems for dayna,ic loading of structures for a state variable description of material behaviour " Inst. of physics, Conf. Ser.No 47 Chap. 1. (1979).

10 R.T. ROCKAFELIAR " Integrals which are convex functions" Pac.J.Math. Vol. 24. No. 3, (1968).

11 F.SIDOROFF "Variables internes en viscoelasticite, I: variables internes scalaires et tensorielles", J. de mécanique, Vol. 14, No. 3 (1975).

12 J.M. THOMAS "Sur l'analyse Numérique des methodes d'éléments finis hybrides et mixtes " Thèse, Paris VI, (1977).