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ANALYSIS OF STEADY STRESSES IN ROTATING ANISOTROPIC DISCS

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ABSTRACT

A full closed-form analysis, based upon the Filonenko-Borodich small displacement technique, for the stress distribution within rotating, constant-thickness, annular, anisotropic discs, is given. The solutions obtained are much more general than those arising from the use of the very specific orthotropic condition, which is misleadingly described as anisotropic by many authors. However, in order to test the validity of this technique stress distributions for the orthotropic case ($\frac{\nu_\theta}{E_\theta} = \frac{\nu_r}{E_r}$) have been computed with its aid. These results are compared favourably with those obtained by previous authors, and highlight the influence of the radius ratio, anisotropic constant, and Poisson's ratio upon the stress distribution.

INTRODUCTION

Thin elastic annular discs are a design feature of many industrial systems such as gas-turbines, steam-turbines, compressors, flywheels and computer stores. When such discs are designed to rotate at high speed the possibility that they will burst or distort must be considered. Hence, it is often necessary to determine their deformations and the stress distributions within them.

A considerable body of literature exists for the case where isotropic materials are used for the fabrication of discs. However, much interest has been shown in recent years in the possibility of constructing discs (and other machine components) from composite materials which are arranged in such a manner that the direction of their greatest strength corresponds to the direction of the principal load acting upon the device. In this way it should be possible to construct lightweight components of optimum load carrying capability.

The problem of determining the stress distribution within anisotropic discs rotating at steady speed is a complex one and one which has not yet

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received much attention in the literature. However, Tang (1) investigated the stresses produced in orthotropic discs rotating at speed, while Genta and Gola (2) studied the special case where the disc was constructed of materials which gave a stiffness matrix of a particular form. Ari-Gur and Stavsky (3) considered an annular disc rotating at constant velocity and symmetrically laminated from polar-orthotropic composite layers.

In the present paper, the stress distributions existing within annular rotating discs of narrow section, and constructed from any anisotropic material, have been determined. The effect of the variable parameters, radius ratio, anisotropic constant, and Poisson's ratio, has been analysed for the case of the orthotropic annular disc.

ANALYSIS

The analysis for the present investigation is based upon the infinitesimal theory of elasticity as proposed by Filonenko-Borodich (4) for application to an anisotropic body, and is expressed in polar co-ordinates.

The following further assumptions are made:-

(1) The material is macroscopically homogeneous and cylindrically anisotropic

(2) The stress-strain relationships for the material obey a generalised Hooke's Law.

(3) A condition of plane stress exists.

With the above assumptions, the stress-strain relationships in polar co-ordinates can be expressed as follows:

$$\left.\begin{aligned} \varepsilon_r &= \frac{\sigma_r}{E_r} - \frac{\nu_{\theta r}\,\sigma_\theta}{E_\theta} \\ \varepsilon_\theta &= \frac{\sigma_\theta}{E_\theta} - \frac{\nu_{r\theta}}{E_r} \end{aligned}\right\} \qquad \ldots(1)$$

As for all rotating disc problems, because of rotational symmetry, the strain-displacement relationships reduce to,

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad \varepsilon_\theta = \frac{u}{r} \qquad \ldots(2)$$

and the equilibrium equation can be written as

$$\frac{d}{dr}(r.\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = o \qquad \ldots(3)$$

Equations (1) and (2) can be solved for σ_r and σ_θ in u and substituted into equation (3). This yields the following differential equation.

$$r^2 .. \frac{d^2u}{dr^2} + J.r.\frac{du}{dr} - Lu = Nr^3 \qquad \ldots(4)$$

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where $J = \dfrac{(1 + \nu_{\theta r})E_r - \nu_{r\theta}E_\theta}{E_r}$

$L = E_\theta / E_r$

$N = \dfrac{(\nu_{r\theta}\nu_{\theta r} - 1)}{E_r}\,\rho\omega^2$

The general solution of this equation is given as follows

$$u = A_o\, r^{D_1} + B_o\, r^{D_2} + \frac{N}{(6 + 3J - L)}\, r^3 \qquad \ldots(5)$$

where $D_1 = \dfrac{1}{2}\left\{\left(\dfrac{\nu_{r\theta}\, E_\theta - \nu_{\theta r}\, E_r}{E_r}\right) + \left[\left(\dfrac{\nu_{\theta r}\, E_r - \nu_{r\theta}\, E_\theta}{E_r}\right)^2 + 4\,\dfrac{E_\theta}{E_r}\right]^{\frac{1}{2}}\right\}$

and $D_2 = \dfrac{1}{2}\left\{\left(\dfrac{\nu_{r\theta}\, E_\theta - \nu_{\theta r}\, E_r}{E_r}\right) - \left[\left(\dfrac{\nu_{\theta r}\, E_r - \nu_{r\theta}\, E_\theta}{E_r}\right)^2 + 4\,\dfrac{E_\theta}{E_r}\right]^{\frac{1}{2}}\right\}$

from equations (1) and (2) the stress components are given by

$$\sigma_r = \frac{1}{M}\left\{\frac{-\nu_{\theta r}}{E_\theta}\left[A_o\, r^{D_1-1} + B_o\, r^{D_2-1} + \frac{N}{(6 + 3J - L)}\, r^2\right] - \frac{1}{E_\theta}\left[D_1\, A_o\, r^{D_1-1} + D_2\, B_o\, r^{D_2-1} + \frac{3N}{(6 + 3J - L)}\, r^2\right]\right\}$$

$$\sigma_\theta = \frac{1}{M}\left\{\frac{-\nu_{r\theta}}{E_r}\left[D_1\, A_o\, r^{D_1-1} + D_2\, B_o\, r^{D_2-1} + \frac{3N}{(6 + 3J - L)}\, r^2\right] - \frac{1}{E_r}\left[A_o\, r^{D_1-1} + B_o\, r^{D_2-1} + \frac{N}{(6 + 3J - L)}\, r^2\right]\right\} \qquad \ldots(6)$$

where $M = \left(\dfrac{\nu_{r\theta}\ \nu_{\theta r} - 1}{E_r\ E_\theta}\right)$

The integration constants A_o and B_o are determined from the boundary conditions. For the simple constant thickness, annular, model employed here the disc is free from external forces, hence the boundary conditions at the outer and inner radii are:-

$$\left.\begin{aligned} \sigma_r &= 0 \text{ at } r = a \\ \sigma_r &= 0 \text{ at } r = b \end{aligned}\right\} \qquad \ldots(7)$$

Substituting equation (7) in equations (6) gives

$$A_o = -\frac{N\,(3+\nu_{\theta r})\,(a^2 b^{D_4} - b^2 a^{D_4})}{(6+3J-L)\,(\nu_{\theta r}+D_1)\,(a^{D_3} b^{D_4} - b^{D_3} a^{D_4})}$$

and $$B_o = -\frac{N\,(3+\nu_{\theta r})\,(b^2 a^{D_3} - a^2 b^{D_3})}{(6+3J-L)\,(\nu_{\theta r}+D_2)\,(a^{D_3} b^{D_4} - b^{D_3} a^{D_4})}$$

where $$D_3 = \frac{1}{2}\left\{\left[\frac{\nu_{r\theta}E_\theta - (2+\nu_{\theta r})E_r}{E_r} + \left(\frac{\nu_{\theta r}E_r - \nu_{r\theta}E_\theta}{E_r}\right)^2 + 4\frac{E_\theta}{E_r}\right]^{\frac{1}{2}}\right\}$$

and $$D_4 = -\frac{1}{2}\left\{\left[\frac{(2+\nu_{\theta r})E_r - \nu_{r\theta}E_\theta}{E_r} + \left(\frac{\nu_{\theta r}E_r - \nu_{r\theta}E_\theta}{E_r}\right)^2 + 4\frac{E_\theta}{E_r}\right]^{\frac{1}{2}}\right\}$$

Hence, the general solution, for the radial and tangential stresses in any anisotropic annular disc, is as follows:-

$$\sigma_r = -\frac{1}{M}\left[\left(\frac{\nu_{\theta r}+D_1}{E_\theta}\right)A_o r^{D_3} + \left(\frac{\nu_{\theta r}+D_2}{E_\theta}\right)B_o r^{D_4} + \left(\frac{\nu_{\theta r}+3}{E_\theta}\right)\frac{Nr^2}{(6+3J-L)}\right]$$

$$\sigma_\theta = -\frac{1}{M}\left[\left(\frac{D_1\nu_{r\theta}+1}{E_r}\right)A_o r^{D_3} + \left(\frac{D_2\nu_{r\theta}+1}{E_r}\right)B_o r^{D_4} + \left(\frac{3\nu_{r\theta}+1}{E_r}\right)\frac{Nr^2}{(6+3J-L)}\right]$$

...(8)

Note that when $E_\theta/E_r = 1$ and $\nu_{r\theta} = \nu_{\theta r} = \nu$ equation (8) reduces to the isotropic material solution.

For orthotropic homogeneous annular discs, that is where $\nu_{r\theta}/E_r = \nu_{\theta r}/E_\theta$ the stress distribution reduces to the following non-dimensional form

$$\sigma_r = \left[C_1\left[\frac{r}{b}\right]^{(\lambda-1)} + C_2\left[\frac{r}{b}\right]^{-(\lambda+1)} - \left(\frac{\nu_{\theta r}+3}{9-\lambda^2}\right)\left[\frac{r}{b}\right]^2\right]\rho\omega^2 b^2$$

and $$\sigma_\theta = \left[\lambda C_1\left[\frac{r}{b}\right]^{(\lambda-1)} - \lambda C_2\left[\frac{r}{b}\right]^{-(\lambda+1)} - \left(\frac{3\nu_{\theta r}+\lambda^2}{9-\lambda^2}\right)\left[\frac{r}{b}\right]^2\right]\rho\omega^2 b^2$$

...(9)

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$$\text{where} \quad C_1 = \frac{(\nu_{\theta r} + 3)\left[\left(\frac{a}{b}\right)^{(\lambda+3)} - 1\right]}{(9 - \lambda^2)\left[\left(\frac{a}{b}\right)^{2\lambda} - 1\right]}$$

$$C_2 = \frac{(\nu_{\theta r} + 3)\left[\left(\frac{a}{b}\right)^{\lambda-3} - 1\right]\left[\left(\frac{a}{b}\right)^{(\lambda+3)}\right]}{(9 - \lambda^2)\left[\left(\frac{a}{b}\right)^{2\lambda} - 1\right]}$$

$$\text{and} \quad \lambda = \sqrt{E_\theta/E_r}$$

This orthotropic homogeneous solution is similar to that obtained by a number of other authors, for example (1-3), usually as a result of the use of the stress-function technique.

RESULTS

Non-dimensional radial and tangential stress distributions were computed from equation (9), that is, for the very specific orthotropic case, for annular discs with radius ratio (a/b) of 0.1, 0.2, 0.3 and 0.4. For each of these geometries, computations were made for values of the anisotropic constant λ $(\sqrt{E_\theta/E_r})$ = 1.2, 1.5, 2.0, 2.6, and 3.1. The value of Poisson's ratio $\nu_{\theta r}$, which characteristises compression in the radial direction, produced by tension in the tangential direction, was kept constant at 0.3. The computations are illustrated graphically on Figs. 1 to 4.

Fig. 5 illustrates the non-dimensional radial and tangential stress distributions within a disc of fixed geometry (radius ratio a/b = 0.2), and fixed anisotropic constant (λ = 2.5) but with varying Poisson's ratio ($\nu_{\theta r}$ = 0.2, 0.23, 0.25, 0.27, and 0.3).

Finally, Fig. 6 shows the effect of disc geometry (a/b = 0.1, 0.2, 0.3, 0.4 and 0.5) on the non-dimensional radial and tangential stress distributions within discs constructed of a material which has a Poisson's ratio $\nu_{\theta r}$ = 0.3 and an anisotropic constant λ = 3.6.

DISCUSSION

Stress components σ_r and σ_θ have finite values for all radius ratios (a/b) between zero and unity, and for all values of the anisotropic constant except λ = 3.0. However, at λ = 3.0 L'Hospital's rule can be employed to yield finite values for σ_r and σ_θ.

In real discs the shear stresses arising from acceleration or deceleration, or from (say) turbomachine blades carried on the rim, would influence the magnitude and direction of the principal stresses arising in the disc.

However, for the present analysis such shear stresses can be ignored and the principal stresses will be σ_θ and σ_r, the stress in the axial direction being zero for the plane stress case. Hence, from Figs. 1 to 4, it is possible to see that for each disc geometry, the maximum stress arising in the disc is the hoop stress σ_θ at the disc bore corresponding to the lowest value of the anisotropic constant ($\lambda = 1.2$). In fact the hoop stresses for an homogeneous isotropic material ($\lambda = 1.0$) would be greater still. Similarly, it can be seen that some sort of optimum condition arises in each case for an intermediate value of λ, usually for values of λ between 1.5 and 2.0.

For the computed results portrayed by Figs. 1 to 4, although the value of $\nu_{\theta r}$ is kept constant at 0.3, since the material properties related to the hoop and radial directions are linked through the condition $\nu_{\theta r}/E_\theta = \nu_{r\theta}/E_r$, there will be considerable variation in $\nu_{r\theta}$.

From Fig. 5 it can be seen that a variation from 0.2 to 0.3 for the Poisson's ratio of the disc material related to the tangential direction $\nu_{\theta r}$, and a corresponding variation in the Poisson's ratio related to the radial direction $\nu_{r\theta}$, produces almost no variation in the non-dimensional radial and circumferential stress distributions.

Fig. 6 indicates that the ratio of the inner disc radius to the outer disc radius can appreciably affect the stress distribution within the disc. However, a closer scrutiny reveals a fairly constant maximum (hoop) stress at a non-dimensional radius of about 0.8, and that it would only be in discs where the inner radius is more than about 0.4 times the outer radius, that the point of maximum stress shifts to the disc bore. Of course, Fig. 6 is for the specific case where $\lambda = 3.6$, but for other values of λ a similar situation exists but with different values for the critical ratio of a/b.

CONCLUSIONS

A closed form solution based upon the Filonenko-Borodich small displacement technique, for the stress distribution within rotating, constant thickness, annular, anisotropic discs, has been derived. Results obtained with the aid of this solution, for the very specific orthotropic condition, are shown to correspond to solutions obtained by previous authors usually with the aid of the stress function technique. In proposed future work the above full solution will be used for the analysis of a range of anisotropic conditions.

The computed results included in this report indicate that the effect of anisotropy, on the stress distribution within a disc, is considerable. Similarly, there appears to be an optimum value of the anisotropic constant λ, for the minimum value of the principal direct stress acting within a disc.

The effect of varying the value of Poisson's ratio $\nu_{\theta r}$ over the reasonably realistic range of 0.2 to 0.3, has been shown to be slight.

The influence of the ratio of the inner disc radius to its outer radius, on the stress distribution within an orthotropic disc, has been shown to be significant. However, the choice of radius ratio (a/b) is usually based upon the function of the disc and not upon its stress distribution.

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NOMENCLATURE

a	=	inner radius of disc	(m)
b	=	outer radius of disc	(m)
a/b	=	radius ratio	(NON-DIM)
E_r	=	elastic modulus in radial direction	(GN/m^2)
E_θ	=	elastic modulus in tangential direction	(GN/m^2)
r,θ	=	polar co-ordinates	(m, RAD)
u	=	radial displacements	(m)
ρ	=	mass per unit volume of the disc	(kg/m^3)
ω	=	angular velocity of the disc	(RAD/S)
ε	=	strain	(m/m)
λ	=	$\sqrt{E_\theta E_r}$ = anisoptropic constant	(GN/m^2)
ν_θ	=	Poisson's ratio which characterises compression in r - direction for tension in θ direction	(NON-DIM)
σ	=	stress	(N/m^2)

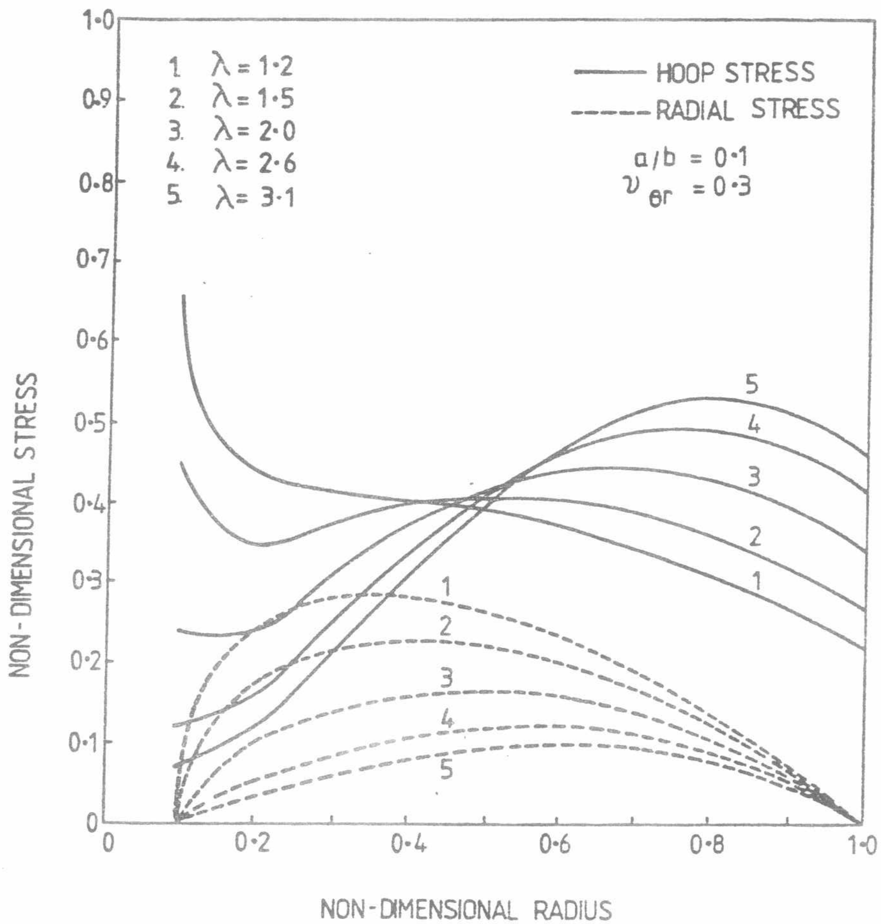

Fig. 1 Stress distributions in a rotating orthotropic disc with radius ratio = 0.1 for various values of the anisotropic constant

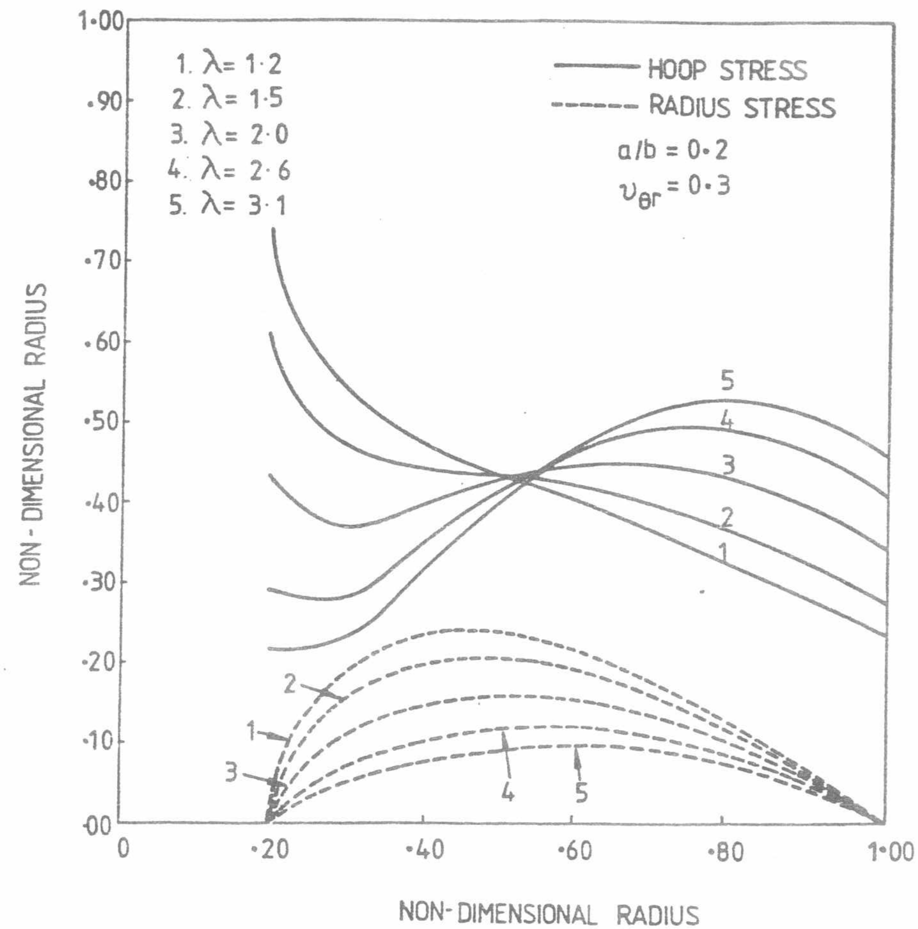

Fig. 2 Stress distributions in a rotating orthotropic disc with radius ratio = 0.2 for various values of the anisotropic constant

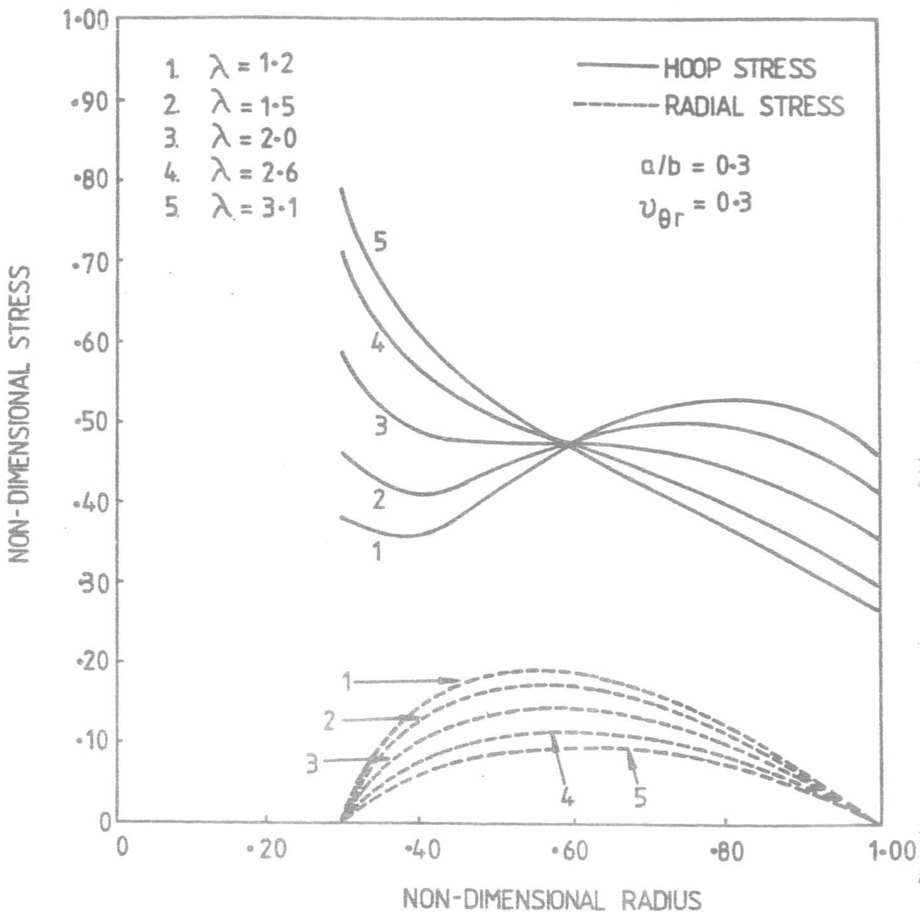

Fig. 3 Stress distributions in a rotating orthotropic disc with radius ratio = 0.3 for various values of the anisotropic constant

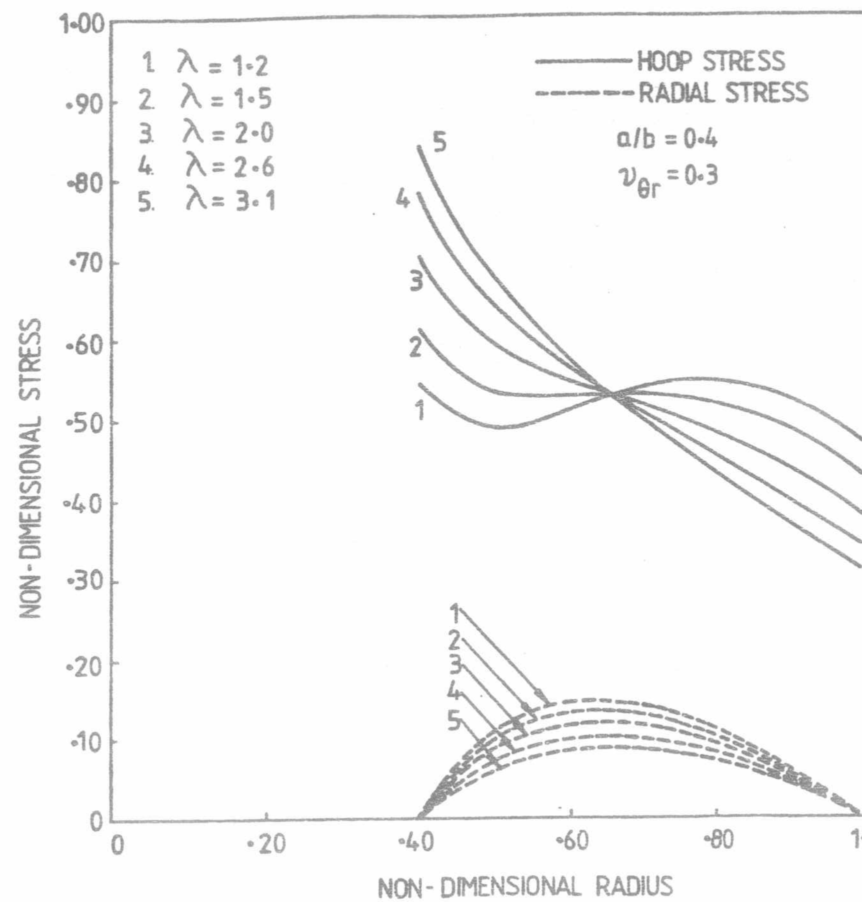

Fig. 4 Stress distributions in a rotating orthotropic disc with radius ratio = 0.4 for various values of the anisotropic constant

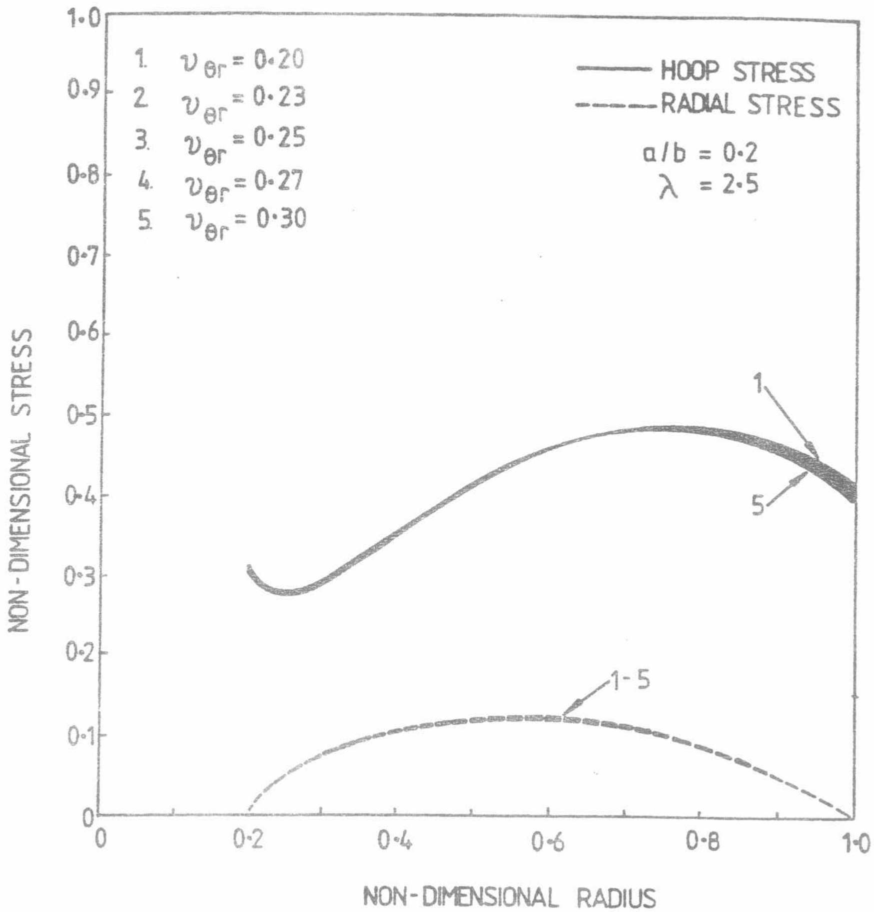

Fig. 5 Stress distributions in a rotating orthotropic disc with radius ratio = 0.2, anisotropic constant = 2.5, and various values of Poisson's ratio

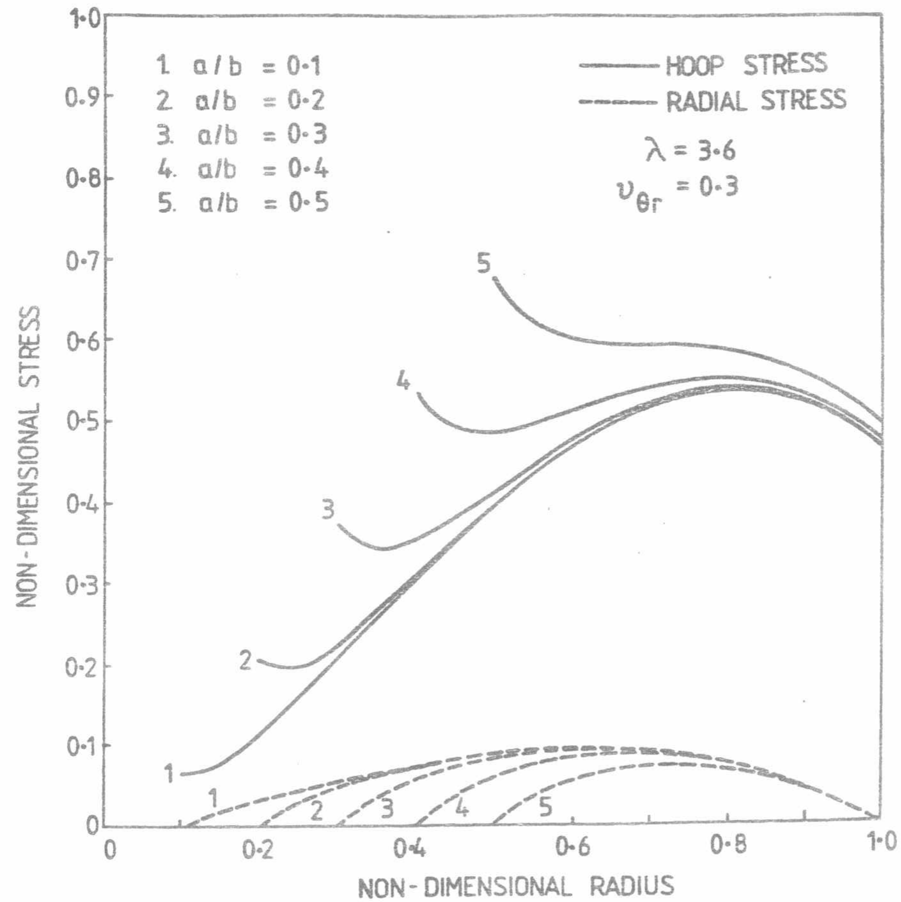

Fig. 6 Stress distributions in a rotating orthotropic disc with anisotropic constant = 3.6 and various values of the radius ratio