A.M.E.

EFFICIENT METHODS FOR THE TRAFFIC ASSIGNMENT PROBLEM

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ABSTRACT

The capacity restrained traffic assignment problem is defined as "assign the vehicular traffic to different links of the network, given an origin-destination matrix for the traffic required between the nodes of the network, and a nonlinear relation for the travel time on each link as function of the traffic flow in that link ". The traffic is assumed to follow user's optimization according to Wardrop's principle, which states that the travel costs on all used pathes for each of the origin-destination pairs are equal and less than the travel cost on the unused pathes.

This paper introduces four methods representing different approachs for solving this problem, the successive incremental assignment, Leblanc's model, a modified form of Leblanc's model and the method of negative loops. These methods are explained and compared with the help of an illustrative example.

1. PROBLEM FORMULATION

The traffic area under study is represented by a directed graph (network), in which each directed arc represents a road direction (link) and each node represents an intersection. Nodes are used as places at which the taffic enters and leaves the network with prespecified amount of flow,

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given in the form of an origin-destination matrix.

It is required to find the traffic flow in each link in the network, assuming that each driver takes the path of least cost from his origin to his destination. According to Wardrop's first principle,[1], an equilibrium state will be reached, which is referred to as "the principle of equal travel costs", and states that : For any two nodes r and s

i- if two or more pathes between node r and node s are actually travelled, then the cost of travel between r and s must be the same for each of these pathes, and

ii-there does not exist an alternative unused path between nodes r and s with less cost than that pathes which are travelled.

It is known that the travel time is a significant factor to the majority of travellers when they choose their routes. We shall use , as it is usual in most traffic models, the travel time as a basic measure of travel cost. We shall use the notation $f_{ij}(x_{ij})$ to represent the average travel time on the link from node i to node j if the flow on that link is x_{ij}.

If all the links of the network are slightly loaded, the average travel time on every link is constant independent of the flow $f_{ij}(x_{ij}) = f_{ij}^{(o)} =$ the length of the link devided by the average speed on that link.

In this case a solution of the traffic assignment problem is simply obtained by solving a shortest path problem between each origin-destination pair. However in most practical situations the average travel time is function of the traffic flow. Many different mathematical forms for this function are proposed in the references, e.g. Branston [2]. A very popular form is given by

$$f_{ij}(x_{ij}) = f_{ij}^{(o)} (1 + (x_{ij} / k_{ij})^n)$$

where $f_{ij}^{(o)}$: The travel time on the link ij at zero flow,

k_{ij} : The capacity of the link ij, at which the average travel time is equal to $2 f_{ij}^{(o)}$, and

n : positive constant.

This form will be used in solving an illustrative example in the next sections. For the simplicity of analysis and without loss of the nonlinearity nature of $f_{ij}(x_{ij})$, we shall take n=2 in the above form.

2. SUCCESSIVE INCREMENTAL ASSIGNMENT [3]

Procedure: 1. Start with the network at zero flow. The travel time on every link ij is $f_{ij}(x_{ij}) = f_{ij}^{(0)}$. Determine a shortest path between every origin-destination pair according to these travel times.

2. A certain predetermined percentage (increment) of the traffic demand (O/D matrix) is added to a shortest path for every origin-destination pair obtained in the last step.

3. Find $f_{ij}(x_{ij})$ for each link, then determine accordingly a shortest path between each origin destination pair.

4. If all the traffic demand is assigned stop, otherwise go to step 2.

Example: Fig.1 shows a simple network

Travel time functions :

$$f_{1,2} = 1 (1 + (x_{1,2}/100)^2)$$
$$f_{1,3} = 2 (1 + (x_{1,3}/200)^2)$$
$$f_{1,4} = 5 (1 + (x_{1,4}/200)^2)$$
$$f_{2,4} = 2 (1 + (x_{2,4}/100)^2)$$
$$f_{3,4} = 2 (1 + (x_{3,4}/200)^2)$$

O/D Matrix :

From Q_1 to S_1 50 , to S_2 100
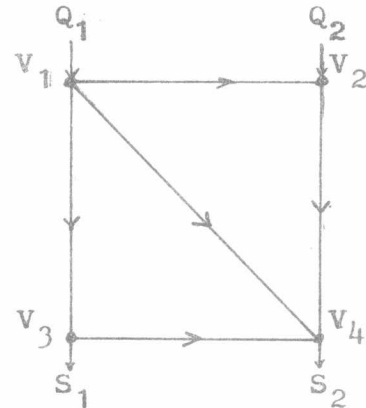From Q_2 to S_1 0 , to S_2 200

Fig.1 Simple network

We suppose that 1% of the traffic will be added in each incrementation step of the procedure.

The traffic flow from Q_1 to S_1 is fixed by the path V_1-V_3 and from Q_2 to S_2 is fixed by the path V_2-V_4 while the traffic from Q_1 to S_2 has 3 alternative pathes .

At zero flow the traffic from Q_1 to S_2 will take the path V_1-V_2-V_4 with shortest time = 3 min. The assignment of the traffic from Q_1 to S_2 to the path V_1-V_2-V_4 will continue in each incrementation and thus the travel time on the links V_1-V_2 and V_2-V_4 will continue to increase until a turning point is reached at which $f_{1,2}(x_{1,2}) + f_{2,4}(x_{2,4}) = f_{1,3}(x_{1,3})+2$. then the traffic from Q_1 to S_2 will change to the path V_1-V_3-V_4. Let this turning point be reached at a percentage "100a" of the total demand, $0 < a \leqslant 1$, then we have:

$$1(1+(100a/100)^2) + 2(1+(300a/100)^2) = 2+2(1+(50a/200)^2) \text{ which gives}$$
$$a = .2301$$

Starting from $a= .24$, $f_{1,3}+f_{3,4}$ will be less than $f_{1,2}+f_{2,4}$ since the additional traffic on the link V_1-V_3 due to traffic from Q_1 to S_1 is less than the additional traffic on the link V_2-V_4 due to the traffic from Q_2 to S_2.

The assignment of the traffic from Q_1 to S_2 to the path $V_1-V_3-V_4$ will continue until $f_{1,2}(x_{1,2})+ f_{3,4}(x_{3,4})= 5$ then it will change to the path V_1-V_4. The value of the percentage "100a" at which the second turn occures is given by

$$2 \left[1+(50a+100a-23/200)^2\right] + 2 \left[1+(100a-23/200)^2\right]=5 \quad \text{which gives} \quad a= .976$$

Thus starting from $a= .98$ till 1 the traffic from Q_1 to S_2 will take the path V_1-V_4.

We note that in the final assignment, the 3 alternative pathes for the traffic from Q_1 to S_2 are all used, while the travel time on these pathes are completely different: for the path $V_1-V_2-V_4$ $f_{1,2}+f_{2,4}= 13$ min,

for the path $V_1-V_3-V_4$ $f_{1,3}+f_{3,4}= 5.04$ min,

and for the path V_1-V_4 $f_{1,4} = 5$ min

We conclude that this method may not realize the principle of equal travel costs, whatever the number of increments used.

3. LEBLAN'S MODEL [4]

Applying the principle of equal travel costs to a traffic network is according to Beckmann, [5], equivalent to solve the nonlinear convex optimization problem: minimize $Z = \sum\limits_{\text{all links } ij} \int_0^{X_{ij}} f_{ij}(t_{ij}) \, dt_{ij}$,

subject to the continuity of flow constraints at every node of the network. Leblan's model is based on this concept and may be summarized in the following procedure:-

1. Start with the network at zero flow, Determine a shortest path between every origin-destination pair according to the travel times $f_{ij}^{(0)}(x_{ij})$.

2. Assign the whole traffic demand (O/D matrix) to the shortest pathes found in step 1. This gives the zero assignment vector $\underline{x}^{(0)}$.
Set iteration counter n=1

3. Determine the new shortest pathes according to the last assignment $\underline{x}^{(n-1)}$.

4. Corrosponding to the new shortest pathes find the traffic assignment $\underline{y}^{(n)}$.

5. Apply a one-dimensional search technique to find the value of $\underline{x}^{(n)}$

$\underline{x}^{(n)} = (1-a_n)\ \underline{x}^{(n-1)} + a_n\ \underline{y}^{(n)}$, $0 \leqslant a_n \leqslant 1$ that minimize

$$Z = \sum_{\text{all links}} \int_0^{x_{ij} \in \underline{x}^{(n)}} f_{ij}(t_{ij})\ dt_{ij}.$$

6. Test the stopping criterion. If it fails set n=n+1 and go to step 3. The stopping criterion may be

- Fixed number of iterations, or

- an upper limit for the max. difference between two successively obtained values of travel times f_{ij} or values of flow x_{ij}, for all links ij.

Solution of the example of fig 1 :

The flow vectors $\underline{x}^{(n)}$ and $\underline{y}^{(n)}$ will be written as a raw vector in the form $(x_{1,2},\ x_{1,3},\ x_{1,4},\ x_{2,4},\ x_{3,4})$ and the travel time vector $\underline{f}$ in the form $(f_{1,2},\ f_{1,3},\ f_{1,4},\ f_{2,4},\ f_{3,4})$.

At zero flow the shortest path from Q_1 to S_2 is $V_1-V_2-V_4$, hence the initial traffic assignment is given by

$\underline{x}^{(0)} = (100, 50, 0, 300, 0)$ and $\underline{f}^{(0)} = (2, 2.125, 5, 20, 2)$.

1st Iteration: According to the above values, the travel time over the path $V_1-V_2-V_4 = 22$ min. A new shortest path from Q_1 to S_2 is given by $V_1-V_3-V_4 = 4$ min. The corrosponding total assigment will be

$$\underline{y}^{(1)} = (0, 150, 0, 200, 100)$$

Apply the search technique

$\underline{x}^{(1)} = (1-a)\ \underline{x}^{(0)} + a\ \underline{y}^{(1)}$

$= (100-100a, 50 + 100a, 0, 300-100a, 100a)$

Substituting in the expression for Z, we obtain

$$Z = \sum_{\text{all links}} f_{ij}^{(0)}\ (x_{ij} + x_{ij}^3 /3\ k_{ij}^2)$$

$= 100 + 100a + 100/6\ a^3 + 100/3\ (1-a)^3 + 100/6\ (.5+a)^3 + 200/3\ (3-a)^3$

To minimize Z we solve dz/da = 0, and obtain a= $8.5 \pm 6.06/2$, i.e. no turning point lies bet 0,1, The min. value of Z lies on the boundary, actually at a = 1

$\underline{x}^{(1)} = \underline{y}^{(1)} = (0, 150, 0, 200, 100)$ and $\underline{f}^{(1)} = (1, 3.125, 5, 10, 2.5)$

2nd Iteration: The travel time over the path $V_1-V_2-V_4 = 1+10=11$ min, over the path $V_1-V_3-V_4 = 3.125+2.5 = 5.625$ min, and over the path $V_1-V_4 = 5$ min. Thus the new assigment from Q_1 to S_2 takes the path

V_1-V_4 and we obtain $\underline{y}^{(2)} = (0, 50, 100, 200, 0)$.

Applying the search technique

$$\underline{x}^{(2)} = (1-a) \underline{x}^{(1)} + a \underline{y}^{(2)}$$

$$= (0, 150-100\ a, 100\ a, 200, 100-100\ a)$$

and $Z = 100\ a + 100/6\ (1.5-a)^3 + 500/12\ a^3 + 100/6\ (1-a)^3$

$dZ/da = 0$ gives $a = 0.244$

$$\underline{x}^{(2)} = (0, 125.6, 24.4, 200, 75.6)$$

and $\underline{f}^{(2)} = (1, 2.7888, 5.0744, 10, 2.2858)$

The travel time over the path $V_1-V_2-V_4 = 11$ min, over the path $V_1-V_3-V_4 = 5.0746$ min and over the path $V_1-V_4 = 5.0744$ min. The traffic from Q_1 to S_2 does not use the path $V_1-V_2-V_4$.

We have already after two iterations an accurate solution which satisfy the principle of equal travel costs.

We have used this model to solve large scale problems and it gives almost accurate results in small number of iterations. A Pascal-program for this model is available in Ref.[6].

4.A MODIFIED FORM OF LEBLANC'S MODEL [9]

Leblanc's model is simplified by excluding the one dimensional search technique and taking $a_n = 1/n$ at the nth iteration, which leads to

$\underline{x}^{(1)} = \underline{y}^{(1)}$, $\underline{x}^{(2)} = 1/2\ \underline{x}^{(1)} + 1/2\ \underline{y}^{(2)}$ ...and in general

$$\underline{x}^{(k)} = 1/k\ (\underline{y}^{(1)} + \underline{y}^{(2)} + .. .+ \underline{y}^{(k)})$$

The proove of the convergence of this method is given by Powell,[7], but actually the convergence is much slower than that of the Leblanc's model.

Solution of the example of fig1

Using the same notations for the flow vector and travel time vector and following the same steps as in section 3, we obtain

$\underline{x}^{(0)} = (100, 50, 0, 300, 0)$, $\underline{f}^{(0)} = (2, 2.125, 5, 20, 2)$

1st Iteration : Shortest path from Q_1 to S_2 $V_1-V_3-V_4$

$\underline{y}^{(1)} = (0, 150, 0, 200, 100)$, $\underline{x}^{(1)} = \underline{y}^{(1)}$ and $\underline{f}^{(1)} = (1, 3.125, 5, 10, 25)$

2nd Iteration : Shortest path from Q_1 to S_2 V_1-V_4

$\underline{y}^{(2)} = (0, 50, 100, 200, 0)$, $\underline{x}^{(2)} = 1/2 (\underline{y}^{(1)} + \underline{y}^{(2)})$,we obtain

$\underline{x}^{(2)} = (0, 100, 50, 200, 50)$ and $\underline{f}^{(2)} = (1, 2.5, 5.3125, 10, 2.125)$

3rd Iteration : Shortest path from Q_1 to S_2 : $V_1-V_3-V_4$

$\underline{y}^{(3)} = (0, 150, 0, 200, 100)$, $\underline{x}^{(3)} = 1/3 (\underline{y}^{(1)} + \underline{y}^{(2)} + \underline{y}^{(3)})$, thus

$\underline{x}^{(3)} = (o, 116.7, 33.3, 200, 66.7)$ and $\underline{f}^{(3)} = (1, 2.681, 5.139, 10, 2.222)$

4th Iteration : Same shortest pathes as in iteration 3, i.e. $\underline{y}^{(4)} = \underline{y}^{(3)}$

$\underline{x}^{(4)} = 1/4 (\underline{y}^{(1)} + \underline{y}^{(2)} + \underline{y}^{(3)} + \underline{y}^{(4)}) = (0, 125, 25, 200, 75)$ and

$\underline{f}^{(4)} = (1, 2.78, 5.078, 10, 2.28)$

The used pathes from Q_1 to S_2 are V_1-V_4 with time 5.078 min and $V_1-V_3-V_4$ with time 2.78 + 2.28 = 5.06 min. This solution is very close to the exact one. However if we try to obtain more accurate solution a very large number of iterations have to be performed.

5. METHOD OF NEGATIVE LOOPS

Sahling,[8], proposed a method for the traffic assignment problem. It depends on the use of incemental networks, which is obtained from the traffic network by adding to the "used" links another links in the opposite direction with negative values of travel times used on that links. Fig.2b shows an incremental network for the network of fig.2a.

The method is based on the fact that if the principle of equal travel times is not reserved in the network, then loops with net negative sum of travel times (negative loops) exist. E.g. in fig.2b two negative loops could be found $V_1-V_4-V_2-V_1$ and $V_1-V_3-V_4-V_2-V_1$ with sums -17 and -17.875 respectively.

To iliminate a negative loop, we add a flow "w" to each of the positive links of the loop and substract the same flow w from each of the negative links. To determine w we have to solve the equation :

$$\sum_{\text{all + links uv}} f_{uv}(x_{uv} + w) = \sum_{\text{all - links uv}} f(x_{uv} - w)$$

The procedure of the method may be summerized as follows:-

1. Start with the network at zero flow. The travel time on every link ij is $f_{ij}(x_{ij}) = f_{ij}^{(0)}$. Determine a shortest path between every origin-destination pair according to these travel times.

2. Assign the whole traffic demand to the network according to the shortest pathes found in the last step.

3. Draw the incremental network and find out the negative loops. If these is no negative loops or there exist negative loops with very small

absolute value of negative sum (beyond a given accuracy limit) stop,
we have obtained the required solution.

4. Take a negative loop with least absolute value of -ve sum. Find w by
solving the above equation, then obtain $w^* = \min(w, \min_{-ve\ links}(x_{uv}))$

The value of w^* represents the max. flow that can be substracted from
the negative links without leading to a net negative flow. A new tra-
ffic assignment is obtained by correcting the flow on links of the
negative loop with w^*.

5. Calculate the new travel times over the network and the corrosponding
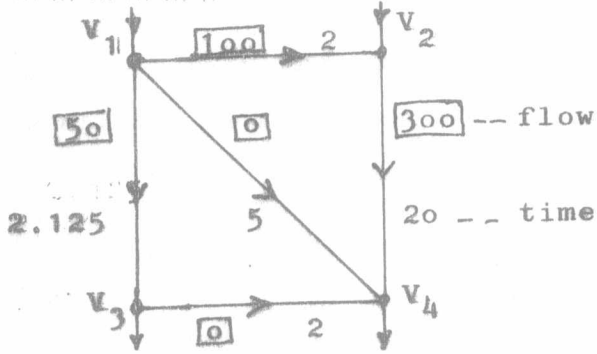shortest pathes. Go to step 2.

Solution of the example of fig 1

Fig 2-a shows the assignment according to the shortest pathes at zero flow.
From fig.2-b, we find that the incremental network contains two negative
loops. We consider the loop with least absolute value of sum of flow, i.e.
the loop V_1-V_4-V_2-V_1. The value of w is obtained by solving the equation
$5(1+(w/200)^2) = (1+(100-w/100)^2) + 2(1+(300-w/100)^2)$, which gives w=149.
$w^* = \min(w, \min(100, 300)) = 100$. Adding a flow 100 to the link V_1-V_4
and substracting the same flow from each of the links V_1-V_2 and V_2-V_4, we
obtain the traffic assignment of fig 2-c. Its incremental network, fig 2-d,
contains only one negative loop V_1-V_3-V_4-V_1. The flow w that corrects the
loop assignment is given by $5(1+(100-w/200)^2)=2(1+(50+w/200)^2)+2(1+(w/200)^2)$
which gives w= 75.6, $w^* = \min(75.6, 100)=75.6$. The resulting total traffic
assignment is shown in fig.2-e, which is the same solution obtained by
Leblancs model in two iterations.

The method of negative loops is an intellegent graphical method that
applies directly the principle of equal travel times without the need of
nonlinear programming formulation. An ALGOL-program for this method is
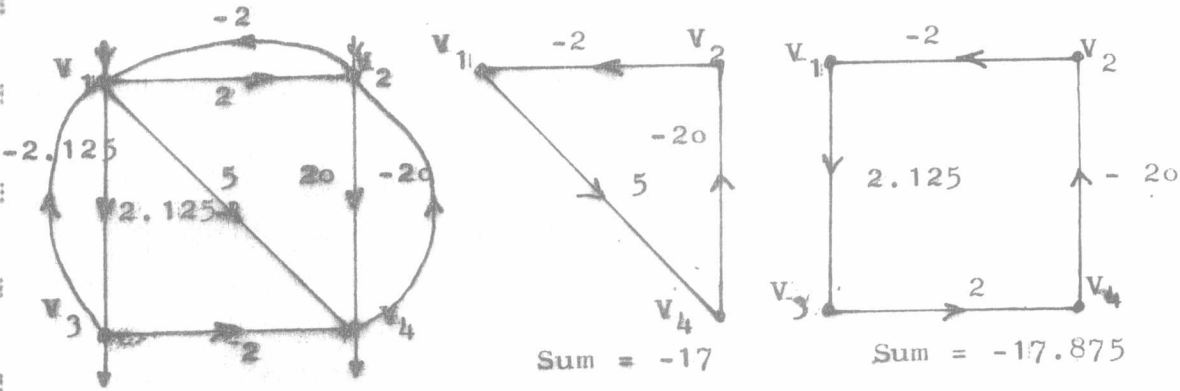given in [8].

6. CONCLUSION

We have considered four models for solving the capacity restrained traffic
assignment problem that represent different approachs for the solution.
These models are briefly explained with the help of an illustrative exa-
mple. The successive incremental assignment method is a heuristic method
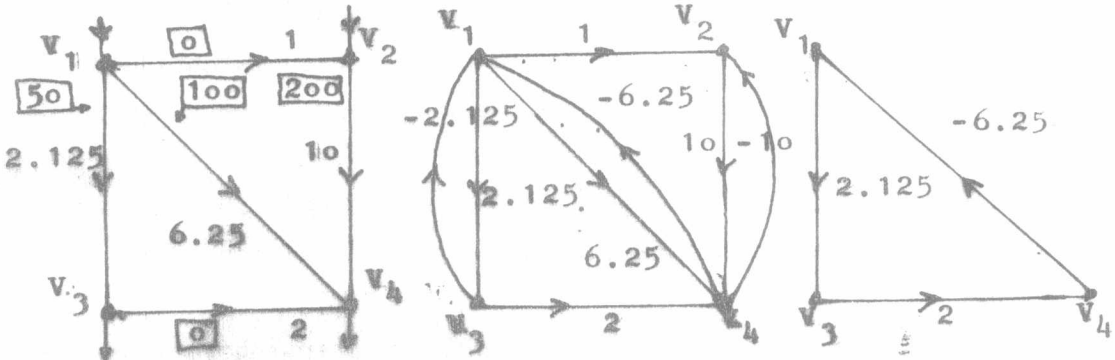that may lead to fault solutions. The model of Leblanc as well as the
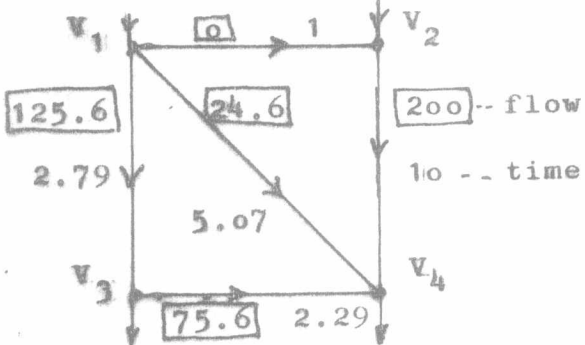
(a) Network

(b) Incremental network and its negative loops

Sum = -17

Sum = -17.875

(c)

(d)

(e)

Fig.2 Solution of example 1 by negative loops

method of negative loops of sahling leads to an exact solution in a small number of iterations. The modified form of Leblanc is much simpler but it coverges very slowly to the exact solution.

7. REFERENCES

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