MILITARY TECHNICAL COLLEGE

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LATERAL VIBRATION OF CANTILEVERS ON VISCOELASTIC FOUNDATIONS

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ABSTRACT

Lateral vibrations of uniform cantilevers resting on viscoelastic foundations are investigated. Effect of foundation parameters and a follower force applied at the cantilever free end, on natural frequencies and vibration stability is studied. The system governing equations reduced to a dimensionless form, is solved using variational method, with three modes of vibrations taken into account. The second and third natural frequencies are shown to increase with the increase of either the foundation elastic and shear rigidities or the applied external tensile force. The first natural frequency increases with increasing the foundation elastic rigidity, while it decreases with the increase of the foundation shear rigidity and/or the tensile follower force, till it vanishes indicating vibration instability. Regions of stable vibrations are determined when the cantilever is subjected to a tensile or compressive follower force. Results obtained can be used for design purposes to determine the foundation parameters that should be guaranteed for stable vibrations when a certain follower force is applied to the beam, or alternatively to determine the limits to be imposed on this force when the cantilever is mounted on a viscoelastic medium of known parameters.

INTRODUCTION

Lateral vibrations of cantilevers resting on viscoelastic foundations might, under certain circumstances, increase indefinitely with time elapse. Smith and Herrmann |1| discussed this problem, and showed that unstable vibrations occur when an externally applied compressive follower force to the cantilever is increased above a certain value, that depends on the system parameters. Wahed |2| verified this conclusion theoretically by solving the system governing equation, then determining the thresholds of vibration instability. In developing the governing equation, he considered the foundation reaction on the beam as resulting from infinite number of closely spaced independent springs and dashpots. More appropriate representation of the foundation behaviour at the contact surface would result when assuming some kind of interaction between the postulated infinite elements |3| , or alternatively

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by considering the foundation as a semi-infinite isotropic continuum and solving the equations governing its dynamic behaviour. For developing a simple, but fairly accurate mathematical model in the latter case, simplifying assumptions with respect to displacements were adopted in |4| during investigating the behaviour of pure elastic foundations. Despite being tedious, the latter approach has the merit of evaluating the foundation parameters in terms of the foundation material properties. Starting from the equations of equilibrium of a continuum, the derivation of the equation governing the dynamics of a viscoelastic foundation at the contact surface is presented in an Appendix.

The analysis of lateral vibrations of beams on viscoelastic foundations carried out in |5| showed that simply supported beams are always stable, except for high compressive follower forces. A simplified analysis of cantilevers vibrations, based on a two-mode approximation, revealed that instability may result even in the abscence of external follower forces, due to the assumed interaction between the adjacent elements representing the foundation.

In this paper a more detailed analysis of cantilevers vibrations is carried out, using a three-mode approximation, to investigate the effect of foundation and follower forces on the vibration stability and natural frequencies.

GOVERNING EQUATIONS

The foundation reaction acting on the cantilever is assumed to be q(x,t), as shown in Fig. 1. The equation governing the small transverse vibration of a uniform cantilever of flexural rigidity EI, length ℓ, and mass per unit length m_1 is

$$EI\ \partial^4 v/\partial x^4 + m_1\ \partial^2 v/\partial t^2 = -q(x,t) \qquad (1)$$

Viscoelastic foundation dynamic behaviour at the contact surface is governed by the equation

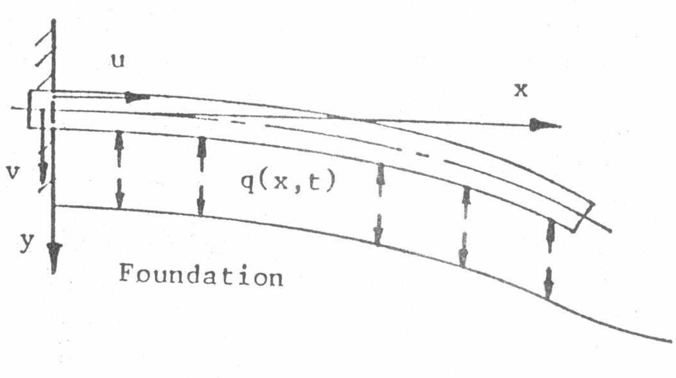
$$s\ \partial^2 v/\partial x^2 - kv + c\ \partial^3 v/\partial t \partial x^2 - b\ \partial v/\partial t - m_o\ \partial^2 v/\partial t^2 + q(x,t) = 0 \qquad (2)$$

where s, k, c, b, and m_o are foundation parameters.

Substituting for q(x,t) from eq.(2) into eq.(1), we get

$$EI\ \partial^4 v/\partial x^4 - s\ \partial^2 v/\partial x^2 + kv - c\ \partial^3 v/\partial t \partial x^2 + b\ \partial v/\partial t + m\ \partial^2 v/\partial t^2 = 0 \qquad (3)$$

where $m = m_o + m_1$



Using the following reference quantities:

$\ell_r = \ell$, $t_r = \ell^2(m/EI)^{\frac{1}{2}}$, $s_r = EI/\ell^2$, $k_r = EI/\ell^4$, $c_r = (mEI)^{\frac{1}{2}}$, $b_r = (mEI)^{\frac{1}{2}}/\ell^2$

eq.(3) can be rewritten in the dimensionless form

$$\partial^4V/\partial X^4 + KV - S\,\partial^2V/\partial X^2 + B\,\partial V/\partial T - C\,\partial^3V/\partial T\partial X^2 + \partial^2V/\partial T^2 = 0 \qquad (4)$$

where $V = v/v_r$, $X = x/\ell_r$, $K = k/k_r$, $S = s/s_r$, ... etc.

The boundary conditions are:

$$V = 0 = \partial V/\partial X \text{ at } X = 0 \text{ , and } \partial^2V/\partial X^2 = 0 = \partial^3V/\partial X^3 \text{ at } X = 1 \qquad (5)$$

SOLUTION OF EQUATION OF MOTION

The variational form of eq.(4) is

$$\int_0^1 (\partial^4V/\partial X^4 + KV - S\,\partial^2V/\partial X^2 + B\,\partial V/\partial T - C\,\partial^3V/\partial T\partial X^2 + \partial^2V/\partial T^2)\,\delta V = 0 \qquad (6)$$

The solution of this equation can be assumed in the form

$$V = \sum_{i=1}^{n} \phi_i(X)\psi_i(T)$$

where ϕ_i is a function satisfying the boundary conditions at X=0 and X=1, while ψ_i is a yet unknown normal coordinate.

For the j^{th} virtual displacement δV_j , eq.(6) transforms to

$$\int_0^1 \Big(\sum_{i=1}^{n} d^4\phi_i/dX^4\,\psi_i + K\sum_{i=1}^{n}\phi_i\psi_i - S\sum_{i=1}^{n} d^2\phi_i/dX^2\,\psi_i + B\sum_{i=1}^{n}\phi_i d\psi_i/dT$$
$$- C\sum_{i=1}^{n} d^2\phi_i/dX^2 . d\psi_i/dT + \sum_{i=1}^{n}\phi_i d^2\psi_i/dT^2\Big)\,\phi_j\delta X = 0 \text{ , } j=1,2,\dots,n \qquad (7)$$

Changing the order of integration and summation, we get

$$\sum_{i=1}^{n} (\alpha_{ij}d^2\psi_i/dT^2 + \beta_{ij}\,d\psi_i/dT + \gamma_{ij}\psi_i) = 0 \qquad \text{, } j=1,2,\dots,n \qquad (8)$$

where $\alpha_{ij} = \int_0^1 \phi_i\phi_j dX$, $\beta_{ij} = \int_0^1 (B\phi_i - C\,d^2\phi_i/dX^2)\phi_j dX$, and

$\gamma_{ij} = \int_0^1 (d^4\phi_i/dX^4 + K\phi_i - S\,d^2\phi_i/dX^2)\phi_j dX$

Taking ϕ_i to be the i^{th} characteristic function of the undamped free vibrations of the cantilever alone, and from the orthogonality of vibration modes, eq.(8) takes the form

$$\alpha_{jj}d^2\psi_j/dT^2 + B\,\alpha_{jj}d\psi_j/dT - C\sum_{i=1}^{n}\delta_{ij}d\psi_i/dT + (\lambda_j^4\alpha_{jj} + K\alpha_{jj})\psi_j$$
$$- S\sum_{i=1}^{n}\delta_{ij}\psi_i = 0 \qquad \text{, } j=1,2,\dots,n \qquad (9)$$

where $\delta_{ij} = \int_0^1 d^2\phi_i/dX^2 . \phi_j dX$, and λ_j^2 is the eigen value of the j^{th} vibration mode.

Vibration modes are thus coupled through the terms containing $\sum_{i=1}^{n}\delta_{ij}$.

considering three modes of vibration. The functions $\phi_i(X)$, as given in |6|, yield:

$\alpha_{11} = \alpha_{22} = \alpha_{33} = 1$, $\delta_{11} = 0.85824$, $\delta_{12} = 1.8738$, $\delta_{13} = 1.5645$, $\delta_{21} = -11.74$
$\delta_{22} = -13.294$, $\delta_{23} = 3.229$, $\delta_{31} = 27.453$, $\delta_{32} = -9.042$, $\delta_{33} = -45.904$,

and $\lambda_1^2 = 3.516$, $\lambda_2^2 = 22.0346$, $\lambda_3^2 = 61.697$

With the solution of eq.(9) assumed in the form $\psi_j = A_j e^{pt}$, the system characteristic equation is

$$\det\{d_{ij}\} = 0 \qquad (10)$$

where $d_{ij} = p^2 + (B - C\delta_{ij})p + \lambda_i^4 + K - S\delta_{ij}$, for $i=j$

$= -(Cp + S)\delta_{ji}$, for $i \neq j$

and the determinant $\det\{d_{ij}\}$ is of the third order.

Setting both B and C equal to zero, and substituting $p=j\omega_n$ in eq.(10), then solving the resulting frequency equation, the system first three natural frequencies are obtained. The effect of foundation elastic and shear rigidities on the natural frequencies is depicted in Fig.2. The figure shows that while both ω_{n2} and ω_{n3} increase with the increase of K and/or S, the first natural frequency increases with K, and decreases with the increase of S. The variation of ω_{n2} and ω_{n3} is shown to be more sensitive to the increase of S than to the increase of K. At certain values of S, the first natural frequency vanishes, indicating vibrations instsbility. These results conform qualitatively with those predicted with the two-mode approximation presented in |5|.

It is to be noted that when an external tensile or compressive follower force f , or in a dimensionless form $F=f\ell^2/(EI)$, is applied to the beam, it can simply be accounted for by adding it to, or subtracting it from S, respectively. This is because this force is entered in the equation of motion; i.e. eq.(4), as $\mp F\, \partial^2 V/\partial X^2$. Thus, with applied external follower force, the parameter S is replaced in eq.(10) by $F_1 = S+F$, where F is positive for tensile forces, and negative for compressive ones.

Now, if a cantilever resting on an elastic foundation is subjected to a tensile follower force, this force should not exceed certain limit in order to avoid vibration instability. The variation of the critical value of F_1 ; namely F_{1c} , with K is shown in Fig.3. An emperical relation that fits the curve well, with an error less than 3% occuring at K=200, is

$$F_{1c} = 32.2 + 7.25\, K^{0.8} \qquad (11)$$

This relation can be made use of to determine the maximum allowable tensile follower force F_c subjected to the beam, provided that K and S are known.

Vibration stability, when damping is present, can be determined by applying Routh's criterion to eq.(10). The obtained results for different combinations of foundation parameters and tensile follower forces are presented in Fig.4, which shows the regions of stable vibrations of the cantilever. For small values of C ($C_s \leq 0.1$), the foundation elastic rigidity is seen to have negligible effect on the stability boundary. The critical value of the linear damping coefficient B

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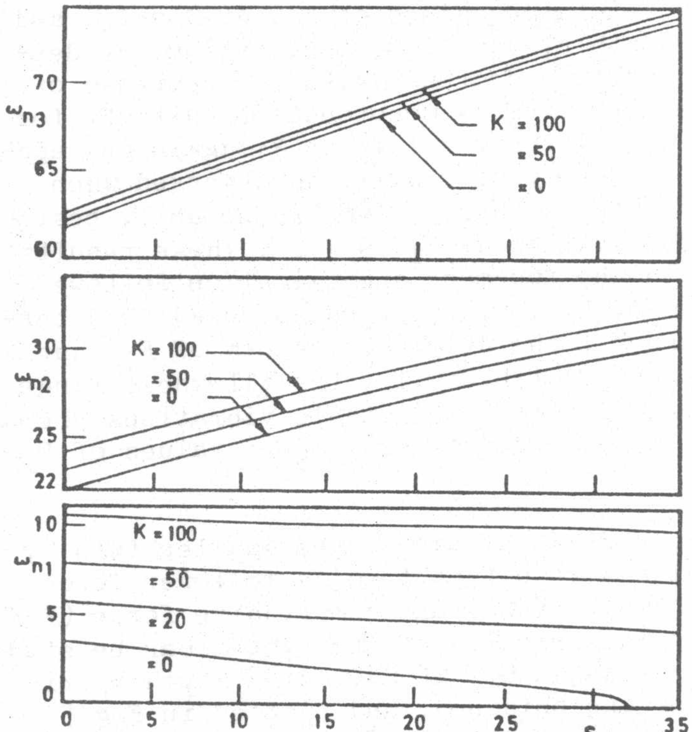

Fig. 2 – Variation of Natural Frequencies with Foundation Elastic and Shear Rigidities

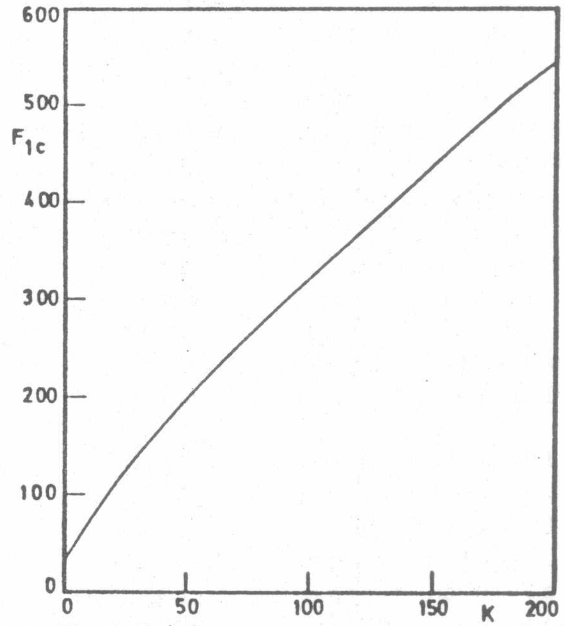

Fig. 3 – Variation of The Critical Resultant Tensile Follower Force with K

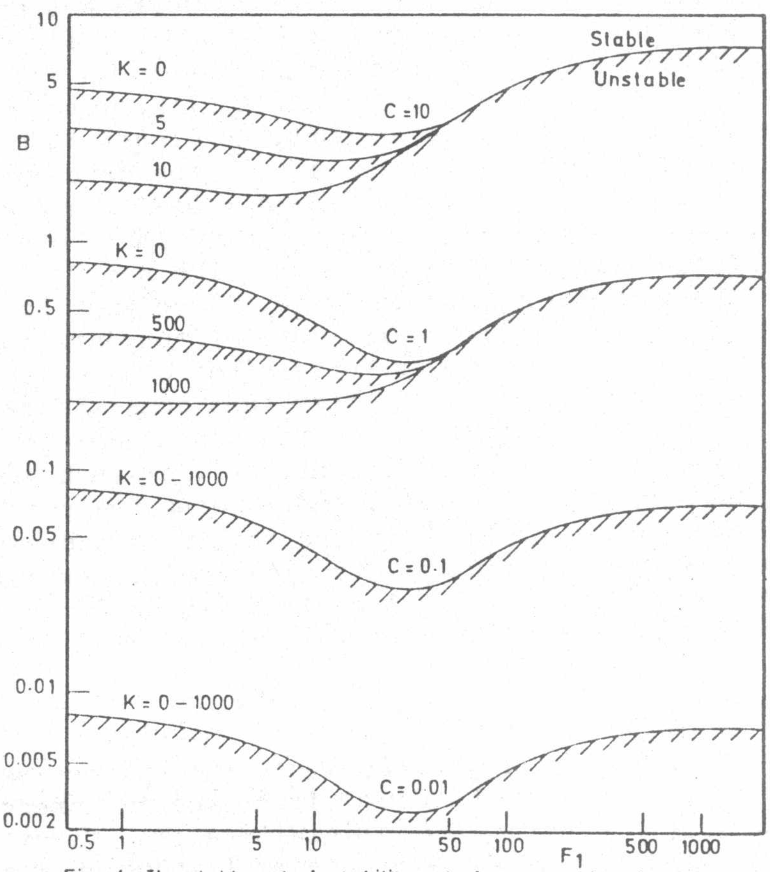

Fig. 4 – Thresholds of Instability of Lateral Vibrations when Resultant Tensile Follower Force is Applied to The

increase of F_1 , till it attains a minimum at $F_1 \simeq 32$. It then increases to reach asymptotically a value of $0.72C_s$ at large values of F_1. At C = 1 , and for $F_1 \lessdot 40$, B_c is shown to depend on both F_1 and K. In this region, B_c decreases with the increase of K, and is seen to decrease with the increase of F_1 for the smaller values of K, while it increases monotoneously with F_1 for K >1000 . When F_1 is greater than 40 , B_c depends solely on F_1 again and approaches asymptotically 0.72. The same tendency of results is observed when C=10 , except that the dependence of B_c on K, when $F_1 <40$, is notable, beside that the final value of B_c at the higher values of F_1 is 7.2 . These results are different from those predicted with the two-mode approximation followed in |5|, specially at the higher values of F_1 . Figure 4 can be used to determine, for each combination of B, C, and K , the allowable values of F_1 that render the vibrations stable. For instance, with C=0.1 , the allowable range of F_1 is $8<F_1<110$, for B=0.05 . Generally, for $F_1>2$, stable vibrations are guaranteed when B/C>0.72 . This limit is extended towards lower values of F_1 when both C and K are increased.

If the cantilever is subjected to compressive follower force smaller than S, the same previous conclusions hold, with a resultant tensile follower force of value (S-F). On the other hand, if F>S, a resultant compressive force (F-S) is applied to the beam. The dynamic characteristics of the system can be studied in this case using eq.(10) with S replaced by -(F-S). Using Routh's criterion, conditions of stable vibrations are obtained, and plotted in Fig.5 . The figure gives the allowable values of the resultant compressive force for the different combinations of K, B, and C. With C=0, this force is constant when B=0, and increases linearly with K when B>0. At any value of K, the allowable force increases with the increase of B. It is worth noting that these conclusions conform well with those reported in | 2 | , where the analysis considered two modes of vibrations, the maximum deviation in the values obtained being less than 4% in the range of K and B dealt with. This indicates that if more than three modes were taken into account, no significant improvement in results would be expected. For the values of C>0, the critical force increases with the increase of either K or B. The increase of C would allow the application of higher compressive follower forces only when both K and B are large (e.g. B=5 and K>120) , but at the low values of B and K , the increase of C is detrimental regarding the allowable force, since this force is reduced. The figure shows also that at certain combinations of foundation parameters, the maximum allowable compressive follower force might attain zero or even negative values. For these combinations, no resultant compressive force would be applied to avoid vibration instability. In this case, the critical value of foundation viscous linear damping coefficient; B_c , at each value of C and K can be obtained by setting S=0 in eq.(10), and finding the conditions under which sustained oscillations occur. The obtained results are presented in Fig.6 . It is to be noted that for foundations with $B<B_c$, unstable vibrations might be stabilized by applying resultant tensile follower force to the beam, the value of which can be determined from Fig.4 .

CONCLUSION

Analysis of lateral vibrations of cantilevers resting on viscoelastic foundations showed that the second and third natural frequencies increase with the increase of foundation elastic and/or shear rigidities. The first natural frequency is shown to increase with the increase of foundation elastic rigidity and decrease of its shear rigidity. At certain value of shear rigidity, depending on elastic rigidity, the first natural frequency

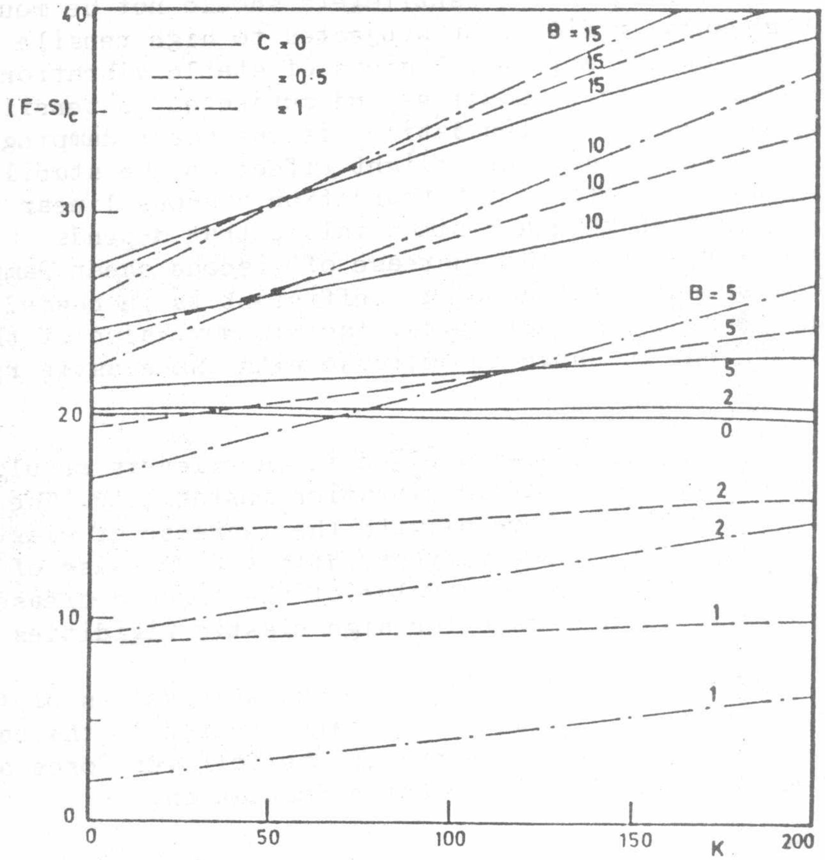

Fig. 5-Variation of Critical Resultant Compressive Follower Force, (F-S)$_c$, with Foundation Parameters

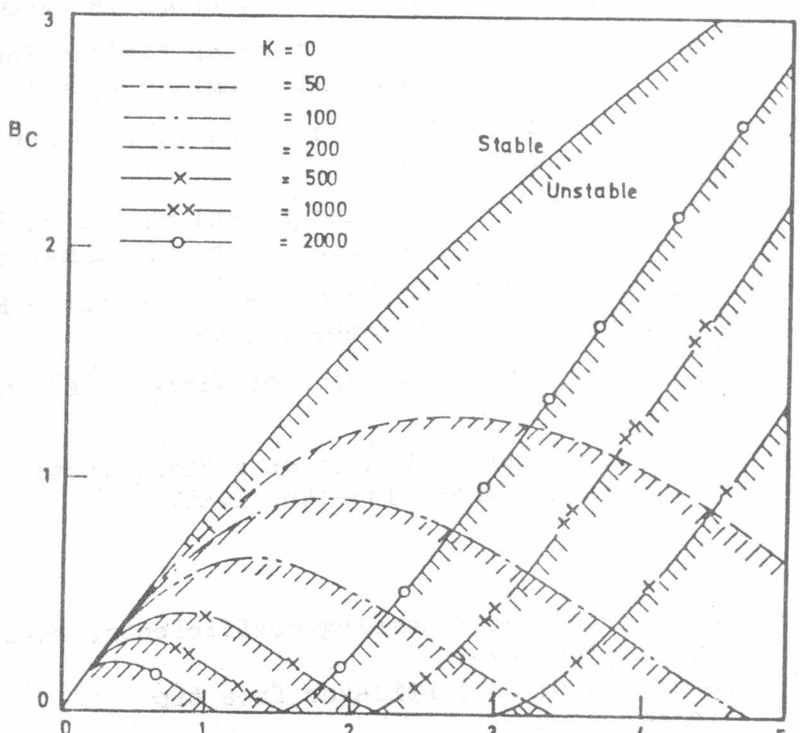

Fig. 6-Lower Limits of B Corresponding to Each Combi-

vibration instability. Consequently, cantilevers should not be mounted on foundations with high shear rigidity or subjected to high tensile follower forces to avoid vibration instability. Regions of stable vibrations of cantilevers mounted on viscoelastic foundations and subjected to tensile follower forces, are determined. For small foundation viscous shear damping coefficient, elastic rigidity is shown to have very slight effect on the stability boundary. To guarantee stable vibration, the foundation viscous linear damping coefficient should be higher than a minimum value, that depends on the resultant tensile follower force. With the increase of viscous shear damping coefficient, the minimum viscous linear damping coefficient is increased. At low values of resultant tensile follower force, the minimum value of the viscous linear damping coefficient is shown to decrease with the elastic rigidity increase.

Resultant compressive follower forces applied to cantilevers should not be increased above certain values to avoid vibration instability. The higher limit of this force is shown to increase with the increase of elastic rigidity and/or viscous linear damping coefficient. With the increase of the viscous shear damping coefficient, the limiting value of the force decreases for small elastic rigidities, while it increases for high elastic rigidities.

Presented results can be used to determine the suitable values of the foundation parameters, when certain follower forces are applied to the cantilever, or alternetively to find the limits imposed on the follower force applied to a cantilever mounted on a certain viscoelastic foundation.

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NOMENCLATURE

- b, c foundation viscous linear and shear damping coefficients, respectively
- E modulus of elasticity of cantilever
- f external follower force acting on cantilever free tip
- f_1 resultant follower force
- I moment of inertia of cantilever
- k foundation elastic rigidity

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m_o, m_1, m foundation, cantilever, total mass per unit length, respectively

q reaction between cantilever and foundation

s foundation shear rigidity

t time

u, v displacements in the x and y directions, respectively

x, y coordinates

$\alpha, \beta, \gamma, \delta$ coefficients

$\omega_{n1}, \omega_{n2}, \omega_{n3}$ first, second, third natural frequencies, respectively

ϕ cantilever lateral deflection

ψ normal coordinate

$b_r, c_r, \ldots$ reference quantities

B, C, K,... dimensionless parameters and variables

Subscript

c critical

APPENDIX

The set of equations governing the dynamic behaviour of an infinite visco-elastic body in two-dimensional deformation, are obtained by applying Newton's laws to an infinitismal rectangular element, of dimensions dx and dy, In the abscence of body forces, these equations are:

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = \rho\ddot{\bar{u}} \quad \text{and} \quad \partial\sigma_y/\partial y + \partial\tau_{xy}/\partial x = \rho\ddot{\bar{v}} \qquad (a1)$$

where σ_x and σ_y are normal stresses, τ_{xy} is shear stress, $\bar{u}$ and $\bar{v}$ are displacements of the C.G. of the element in the x and y directions, respectively, and $\ddot{\bar{u}}$ and $\ddot{\bar{v}}$ are the respective accelerations.

The stress-strain relations are

$$\sigma_x = E(\partial\bar{u}/\partial x + \mu\,\partial\bar{v}/\partial y)/(1-\mu^2) + 2\eta(2\,\partial\dot{\bar{u}}/\partial x - \partial\dot{\bar{v}}/\partial y)/3$$

$$\sigma_y = E(\partial\bar{v}/\partial y + \mu\,\partial\bar{u}/\partial x)/(1-\mu^2) + 2\eta(2\,\partial\dot{\bar{v}}/\partial y - \partial\dot{\bar{u}}/\partial x)/3$$

$$\tau_{xy} = 0.5E(\partial\bar{v}/\partial x + \partial\bar{u}/\partial y)/(1+\mu) + \eta(\partial\dot{\bar{v}}/\partial x + \partial\dot{\bar{u}}/\partial y)$$

where E, μ, and η are the modulus of elasticity, Poisson's ratio, and viscosity index of the body material, respectively.

The variational form of the second of eqs.(a1) is:

$$\int_{V}(\partial\sigma_y/\partial y + \partial\tau_{xy}/\partial x - \rho\ddot{\bar{v}})\,\delta\bar{v}\,dV = 0 \qquad (a2)$$

where V is the volume .

The displacements $\bar{u}$ and $\bar{v}$ can be assumed in the form

$$\bar{u} = \sum_{i=1}^{M} U_i(x,t)H_i(y) \quad \text{and} \quad \bar{v} = \sum_{j=1}^{N} V_j(x,t)R_j(y) \qquad (a3)$$

where $H_i(y)$ and $R_j(y)$ are functions satisfying certain conditions along the lines y = constant.

Now,

$$\delta\bar{u} = \sum_{i=1}^{M} \delta U_i H_i \quad \text{and} \quad \delta\bar{v} = \sum_{j=1}^{N} \delta V_j R_j$$

Considering any n^{th} virtual displacement, eq.(a2) can be rewritten as

Integrating the first term by parts w.r.t. y, we get

$$\int_{V}\{(\partial\tau_{xy}/\partial x - \rho\ddot{v}) - \sigma_y R_n'\}\ \delta V_n\, dV + \int_A |\sigma_y R_n|_0^y\ \delta V_n\, dA \doteq 0$$

where $R_n' = dR_n/dy$ and $H_n' = dH_n/dy$

Using the stress-strain relations together with eq.(a3), the last equation takes the form:

$$\int_A \Big| \sum_{i=1}^{M} \{(B_{in} - C_{in})\ \partial U_i/\partial x + (D_{in} + F_{in})\ \partial^2 U_i/\partial x\partial t\} + \sum_{j=1}^{N} \{G\ \partial^2 V_j/\partial x^2 - K_{jn} V_j + L_{jn}\partial^3 V_j/\partial x^2\partial t - Q_{jn}\partial V_j/\partial t - S_{jn}\partial^2 V_j/\partial t^2\} + W_{yn}\Big|\ \delta V_n\ dA = 0 \qquad (a4)$$

where n = 1,2,...,N , and

$B_{in} = \int 0.5EH_i' R_n'/(1+\mu)\ dV$, $C_{in} = \int E\mu H_i' R_n'/(1-\mu^2)\ dV$, $D_{in} = \int \eta H_i' R_n\ dV$,

$F_{in} = \int 2\eta H_i' R_n'/3\ dV$, $G_{jn} = \int 0.5ER_j R_n/(1+\mu)\ dV$, $K_{jn} = \int ER_j' R_n'/(1-\mu^2)\ dV$,

$L_{jn} = \int \eta R_j R_n\ dV$, $Q_{jn} = \int 4\eta R_j' R_n'/3\ dV$, $S_{jn} = \int \rho R_j R_n\ dV$, $W_{yn} = |\sigma_y R_n|_y$

For $\delta V_n \neq 0$, the integrand in eq.(a4) should be zero.

When M and N are increased to infinity, and on the flat surface of the medium; namely at y=0 , where the external force q(x,t) is applied, and where the displacement $\bar{u}(x,t)$ can be assumed zero |7|, it can be shown that eq.(a4) is the variational form of the partial differential equation

$$s\ \partial^2 v/\partial x^2 - kv + c\ \partial^3 v/\partial x^2\partial t - b\ \partial v/\partial t - m_o\ \partial^2 v/\partial t^2 + q(x,t) = 0$$

where

$s = G_{jn}$, $k = K_{jn}$, $c = L_{jn}$, $b = Q_{jn}$, and $m_o = S_{jn}$

For pure elastic medium, both b and c are zero. The resulting equation in this case is identical to that derived in |4|.