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ANALYSIS OF ANISOTROPIC ELASTIC FOUNDATION UNDER CONCENTRATED FORCES

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ABSTRACT

A detailed analysis for anisotropic semi-infinite medium, under the effect of concentrated forces, is given. The analysis deals with some kind of anisotropy, in which the material properties are cylindrically orthotropic. The effect of a line concentrated load on a semi-infinite cylindrically orthotropic foundation is treated. The radial and tangential displacement components are chosen to satisfy the equilibrium, compatability and boundary conditions of the problem. Using the generalized Hooke's law, the assumed displacement functions leads to a simple radial stress, which occurs in such kind of problems, as found in literature. The effect of anisotropic constant on radial and tangential displacement components is discussed. Variations of displacement components along the radial and tangential directions are given.

INTRODUCTION

The problem of determining the stress distribution within a homogeneous, elastic, half space under the effect of concentrated forces has received much attention in literature. This problem has different engineering applications such as piles and foundation problems in civil engineering, fretting and contact problems in mechanical engineering. Filonenko [2], Timoshenko [6] and other authors presented the analytical solution for the isotropic case. Jung, Chung et al [4] published a close analytical study for a typical two - dimensional fretting problem of isotropic

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materials.

Many investigators attempted to obtain precise analysis based on the elasticity solution [2] , [6] . In these analyses of isotropic materials, Fig.1., two main assumptions are assumed:

1- there is no shearing stress on the area normal to the radius r,
2- Compressive stress σ_r is assumed as :

$$\sigma_r = -\frac{k}{r}\cos\theta$$

where k is a constant.

In recent years, much interest has been shown for anisotropic materials. In some practical cases, the material or the foundation properties are cylindrically orthotropic. In such materials, reinforcement is made or naturally exists along radial and /or tangential directions. The given work presents the analysis of such kind of anisotropic material under the effect of concentrated forces.

Instead of assuming the stress field, the displacement functions are chosen to satisfy the equilibrium, compatability and boundary condition of the problem. Introducing the generalized Hooke's law, the equations of equilibrium are obtained in terms of displacements. The unknown constants in the assumed displacement functions can be determined from the boundary conditions.

The effect of anisotropic constant $\lambda = E_r/E_\theta$ on radial and tangential displacements is discussed, where E_r , E_θ are the radial and tangential elastic moduli. Tables and figures are presented to show the variations of displacement components along the radial and tangential directions.

THEORETICAL ANALYSIS

Consider a homogeneous elastic anisotropic medium bounded by a plane yoz and extending indefinitely down from this plane, Fig. 2. The following assumptions are made :

1- the medium is cylindrically orthotropic about the axis oz,
2- a condition of plane strain exists,
3- the stress-strain relations, for the medium, obey the generalized Hooke's law.

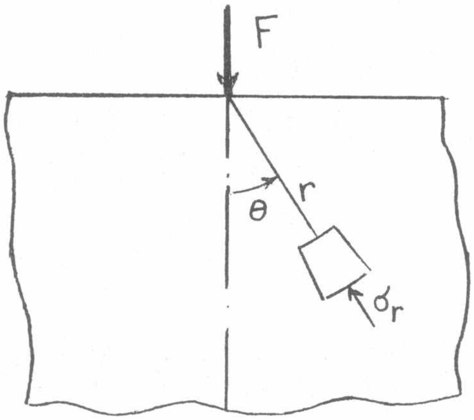

Fig.1. Isotropic half space under a concentrated force.

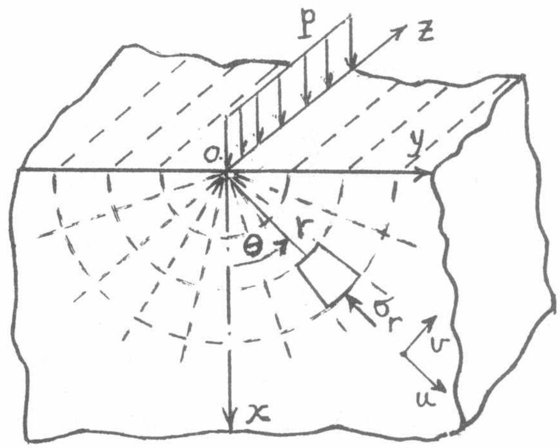

Fig.2. Cylindrically orthotropic half space.

Let a line load of intensity p per unit length acting along the axis oz, normally to the plane yoz.

With the above assumptions, the radial and tangential displacement components u and v, may be assumed as follows :

$$\frac{u}{a}(\rho,\theta) = U_1(\rho)\cos\theta + U_2(\rho)\cos 3\theta, \qquad (1)$$

$$\frac{v}{a}(\rho,\theta) = \alpha_1 U_1(\rho)\sin\theta + \alpha_2 U_2(\rho)\sin 3\theta.$$

where :

- $a = E_\theta/W$: medium characteristic length,
- E_θ : elastic modulus in tangential direction,
- W : specific weight of the medium,
- $\rho = r/a$: dimension-less radial coordinate,
- U_1 and U_2 : unknown displacement functions, depending only on the radial coordinate ρ ,
- α_1 and α_2 : unknown numerical coefficients.

Under a plane strain - condition, the strain - displacement relationships in cylindrical coordinate system is given as:

$$\varepsilon_r = \frac{\partial u}{\partial r} ,$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} , \qquad (2)$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} ,$$

$$\varepsilon_z = \gamma_{rz} = \gamma_{\theta z} = 0.$$

and the stress - strain relationships may be written in the following form, [5] :

$$\sigma_r = \frac{1}{M} (B_{22} \varepsilon_r - B_{12} \varepsilon_\theta),$$

$$\sigma_\theta = \frac{1}{M} (-B_{12} \varepsilon_r + B_{11} \varepsilon_\theta), \qquad (3)$$

$$\tau_{r\theta} = G_{r\theta} \gamma_{r\theta}$$

where :

$$M = B_{11} B_{22} - B_{12}^2 ,$$

B_{11}, B_{22} and B_{12} are the reduced stiffness coefficients defined as :

$$B_{11} = (1- \nu_{rz} \nu_{zr})/ E_r,$$

$$B_{22} = (1- \nu_{\theta z} \nu_{z\theta})/ E_\theta, \qquad (4)$$

$$B_{12} = -(\nu_{r\theta} + \nu_{rz} \nu_{z\theta})/ E_r,$$

In this case of plane strain and due to the fact that $\varepsilon_z = 0$, the lateral stress σ_z can be expressed in terms of σ_r and σ_θ in the form:

$$\sigma_z = \nu_{zr} \sigma_r + \nu_{z\theta} \sigma_\theta \qquad (5)$$

For cylindrically orthotropic material loaded as shown in Fig.2., a simple radial stress is obtained, that means:

$$\sigma_\theta = 0,$$

$$\tau_{r\theta} = 0. \qquad (6)$$

To satisfy conditions (6) and substituting the assumed displacement functions (1) into the strain - displacement relationships and then into the stress-

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strain relations (3), the following equations are obtained :

$$\frac{d\,U_1}{d\,\rho} - \frac{U_1}{\rho}\left(\frac{1+\alpha_1}{\alpha_1}\right) = 0\,,$$

$$\frac{d\,U_2}{d\,\rho} - \frac{U_2}{\rho}\left(\frac{3+\alpha_2}{\alpha_2}\right) = 0\,,$$

$$\frac{d\,U_1}{d\,\rho} - \frac{U_1}{\rho\gamma}(1+\alpha_1) = 0,\qquad (7)$$

$$\frac{d\,U_2}{d\,\rho} - \frac{U_2}{\rho\gamma}(1+3\alpha_2) = 0.$$

where:

$$\gamma = -\,B_{12}/B_{11} \qquad (8)$$

These equations (7) lead to the determination of the coefficients α_1 and α_2 such that :

$$\alpha_1 = -\gamma$$

$$\alpha_2 = \left[(\gamma - 1) \pm \sqrt{1 + 34\gamma + \gamma^2}\,\right]/6 \qquad (9)$$

Neglecting the body forces and under the conditions (5) and (6), the equations of equilibrium, in cylindrical coordinates, are reduced to the following equation.

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r}{r} = 0, \qquad (10)$$

Making use of equations (1), (2) and (3), the radial stress σ_r can be expressed in terms of displacement functions in the form:

$$\sigma_r = \frac{1}{M}\left\{\left[B_{22}\frac{d\,U_1}{d\,\rho} - B_{12}(1+\alpha_1)\right]\cos\theta + \right.$$

$$\left. + \left[B_{22}\frac{d\,U_2}{d\rho} - B_{12}\,(1+3\alpha_2)\right]\cos 3\theta\right\} \qquad (11)$$

For this expression of radial stress to satisfy the equilibrium condition (10), the following two differential equations in displacement functions U_1 and U_2 are obtained :

$$B_{22}\frac{d^2U_1}{d\rho^2}+\frac{1}{\rho}\frac{dU_1}{d\rho}\left[B_{22}-B_{12}(1+\alpha_1)\right]=0,$$

$$B_{22}\frac{d^2U_2}{d\rho^2}+\frac{1}{\rho}\frac{dU_2}{d\rho}\left[B_{22}-B_{12}(1+3\alpha_2)\right]=0. \tag{12}$$

If the expressions of $dU_1/d\rho$ and $dU_2/d\rho$ are substituted from eqs.(7) into eqs(12) they are reduced to the form:

$$\rho^2\frac{d^2U_1}{d\rho^2}-k_1U_1=0,$$

$$\rho^2\frac{d^2U_2}{d\rho^2}-k_2U_2=0. \tag{13}$$

where the coefficients k_1 and k_2 are :

$$k_1=\left(\frac{1+\alpha_1}{\alpha_1}\right)\left[\frac{B_{12}}{B_{22}}(1+\alpha_1)-1\right],$$

$$k_2=\left(\frac{3+\alpha_2}{\alpha_2}\right)\left[\frac{B_{12}}{B_{22}}(1+3\alpha_2)-1\right]. \tag{14}$$

The solution of the two similar - differential equations (13) is given as :

$$U_1=A_1\rho^{g_1}+C_1\rho^{f_1},$$

$$U_2=A_2\rho^{g_2}+C_2\rho^{f_2}. \tag{15}$$

where :

$$g_1=(1+\sqrt{1+4k_1})/2,$$

$$f_1=(1-\sqrt{1+4k_1})/2,$$

$$g_2=(1+\sqrt{1+4k_2})/2,$$

$$f_2=(1-\sqrt{1+4k_2})/2. \tag{16}$$

The quantities A_1, A_2, C_1 and C_2 are four integration constants to be determined from the boundary conditions of the problem.

At indefinite values of the radial parameter ρ , the radial and tangental displacement components tend to zero

$$\text{at } \rho \to \infty : \quad u \to 0 , \quad v \to 0 \tag{17}$$

To satisfy these conditions and keeping in mind that g_1 and g_2 always greater than unity, the first terms on the right hand side of eqs.(15) must be omitted. Hence, the displacement functions (15) are reduced to :

$$U_1 = C_1 \ \rho^{f_1} , \qquad U_2 = C_2 \ \rho^{f_2} . \tag{18}$$

Since the problem is symmetric with respect to the xoz plane, only one half of the medium may be treated $0 \leq \theta \leq \pi/2$. Considering the equilibrium of a half cylindrical part of the medium with unit width along z axis and radius r=a, (i.e. ρ=1), the following two statical conditions must be satisfied.

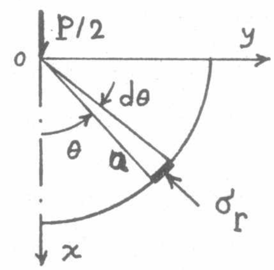

Fig. 3.

$$- \int_0^{\pi/2} \sigma_r \cos \theta \ \ a \, d \theta = P/2 , \qquad \int_0^{\pi/2} \sigma_r \sin \theta \ \ a \, d \theta = 0 \tag{19}$$

Substituting the displacement functions (18) into the normal stress expression (11) and then into the conditions (19) the two integral constants C_1 and C_2 can be obtained.

Once the constants C_1 and C_2 are determined, the displacement functions. hence the displacement components are deduced in the form:

$$u(\rho,\theta) = \frac{2PM}{\pi} \left(\frac{\rho^{f_1}}{N_1} \cos \theta + \frac{\rho^{f_2}}{N_2} \cos 3 \theta \right)$$
$$v(\rho,\theta) = \frac{2PM}{\pi} \left(\alpha_1 \frac{\rho^{f_1}}{N_1} \sin \theta + \alpha_2 \frac{\rho^{f_2}}{N_2} \sin 3 \theta \right) \tag{20}$$

where :

$$N_1 = B_{12} \ (1+\alpha_1) - B_{22} f_1 , \qquad N_2 = B_{12} \ (1+3\alpha_2) - B_{22} f_2 . \tag{21}$$

The radial stress is also deduced as follows :

$$\sigma_r = \frac{-2P}{\pi a} \left(\rho^{f_1 - 1} \cos \theta + \rho^{f_2 - 1} \cos 3 \theta \right) \qquad (22)$$

The isotropic solution is obtained as a special case of the above expressions where $\lambda = E_r/E_\theta = 1$, $\nu_{r\theta} = \nu_{z\theta} = \nu_{rz} = \nu$.

RESULTS AND DISCUSSIONS

Non-dimensional radial and tangential displacement components were computed using expressions (20) and (21) to show how they are affected by the variation of the anistropy.

Table 1. shows the variation of radial displacement u with the radius r, directly under the load $\theta=0$, for different values of anisotropic constant λ = 0.25 , 0.5, 1 and 4

In Table 2., variations of radial displacement u with the angle θ, at radius r = a, and the effect of anisotropy on these variations are given.

Similar values of tangential displacement v were computed and a sample of these values are shown in Figures (4) and (5).

For all values of λ , the radial and tangential displacement components increase rapidly near the point of load application.

Due to the rapid variation of displacement components, semi-logarithmic papers are used to represent these variations. For different values of anisotropic constant, λ = 0.25,1 and 4, Fig. 4. Shows the distribution of the radial displacement, u, along the problem axis of symmetry, θ = 0, and the free surface deformation i.e. tangential displacement v at $\theta = \frac{\pi}{2}$.

Fig. 5. shows the variation of the radial and tangential displacement components, at radius r=a, with the angle θ and the effect of the anisotropy. It may be observed that, in central regions, positive values of radial displacement and negative values tangential displacement occur while these values are reversed near the free surface $\theta > \pi/3$.

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Table 1: Non-dimensional values of radial displacement along the axis of symmetry $\theta=0$, for different values of anisotropic constant $\lambda = E_r/E_\theta$

$\rho = \frac{r}{a}$	$\frac{E_\theta}{P} u(\rho, 0)$				
	$\lambda=1$	$\lambda=1/4$	$\lambda=1/2$	$\lambda=2$	$\lambda=4$
0.025	24.985	1180.86	397.22	11.455	7.303
0.05	13.870	313.04	136.46	6.352	4.050
0.1	7.724	138.73	46.65	3.537	2.254
0.2	4.298	41.32	16.16	1.968	1.255
0.4	2.395	5.68	5.68	1.097	0.699
0.6	1.702	6.25	3.10	0.778	0.497
0.8	1.336	3.81	2.02	0.612	0.390
1.0	1.108	2.60	1.45	0.508	0.321
2.0	0.618	0.82	0.52	0.283	0.180
4.0	0.347	0.27	0.18	0.159	0.099
6.0	0.247	0.12	0.10	0.113	0.072
8.0	0.194	0.08	0.06	0.088	0.055
10.0	0.161	0.06	0.05	0.075	0.047

Table 2: Non-dimensional values of radial displacement along the radius $\rho = 1$, for different values of anisotropic constant λ.

θ	$\frac{E_\theta}{P} u(1, \theta)$				
	$\lambda=1$	$\lambda=1/4$	$\lambda=1/2$	$\lambda=2$	$\lambda=4$
0°	1.107	2.602	1.449	0.5070	0.3210
10°	1.048	2.445	1.373	0.4795	0.3058
20°	0.881	2.012	1.161	0.4035	0.2572
30°	0.645	1.399	0.859	0.2955	0.1884
40°	0.390	0.745	0.531	0.1786	0.1139
50°	0.166	0.185	0.242	0.0761	0.0485
60°	0.011	-0.178	0.038	0.0051	0.0033
70°	-0.058	-0.301	-0.057	-0.0266	-0.0169
80°	-0.051	-0.212	-0.056	-0.0233	-0.0149
90°	0	0	0	0	0

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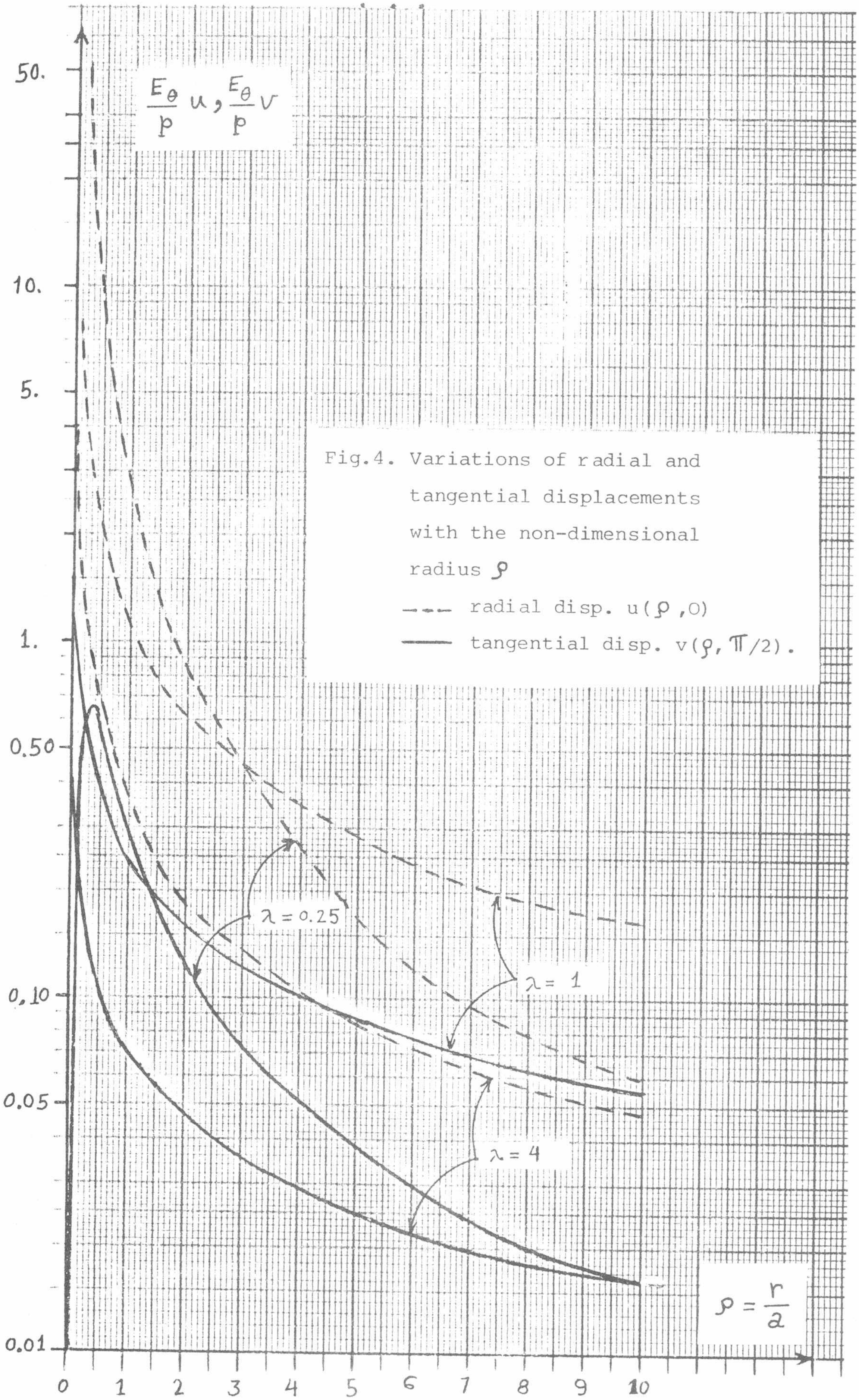

Fig.4. Variations of radial and tangential displacements with the non-dimensional radius ρ

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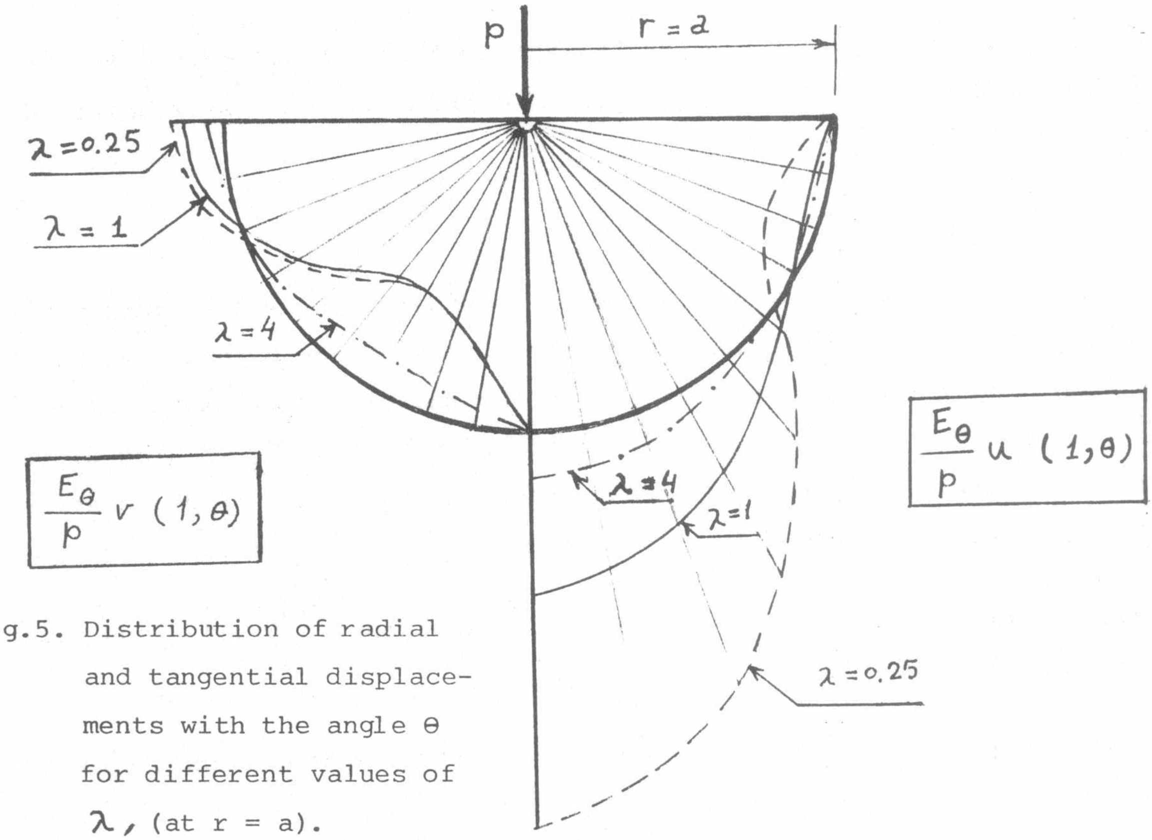

Fig.5. Distribution of radial and tangential displacements with the angle θ for different values of λ, (at r = a).

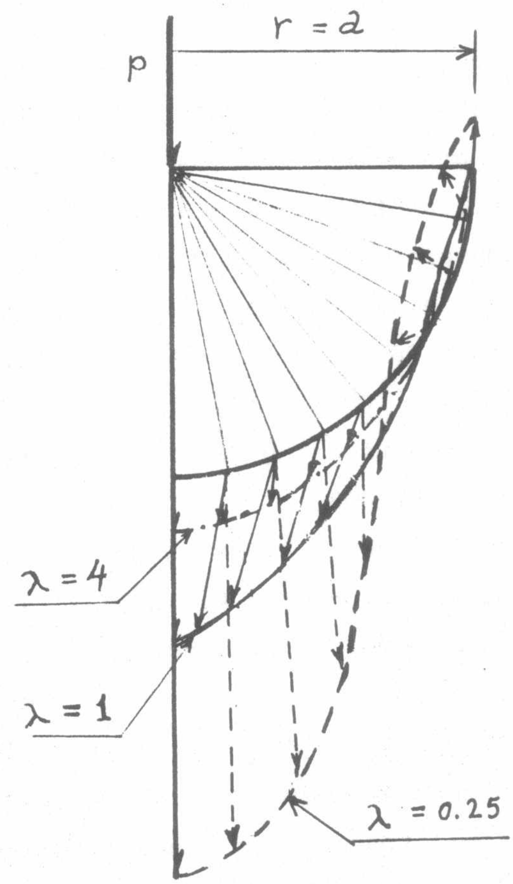

Fig.6. The actual deformed shape of a curved line of radius r=a for different values of λ .

Finally, Fig. 6 . illustrates the resultant actual displacement and the deformed shape of an arc of radius r=a, for different values of anisotropic constant λ = 0.25,1, and 4.

CONCLUSION

The radial and tangential displacement components in a semi-infinite medium which possesses cylindrically orthotropic feature has been analised. The variation of these displacements along the radial and tangental directions are given. The effect of anisotropic constant $\lambda = E_r/E_\theta$ is also discussed. The study deals with a plane strain problem. In a case of plane stress, similar analysis can be done in which $\sigma_z = \tau_{\theta z} = \tau_{rz} \doteq 0$.

This analysis shows that the effect of anisotropy on foundations under concentrated loads is considerable.

An experimental work is proposed to verify the obtained results for both isotropic and cylindrically orthotropic cases.

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