MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

MODELING OF GEAR TOOTH BY THICK PLATE

Kamal Hassan, Dr. Eng.* -Henry Busby, Ass.Prof.**

& Donald Houser, Prof. **

ABSTRACT

The recent approach of Bergan and Wang for the shear inclusion in plate deformation is used in modeling the gear tooth as a cantilever plate. Using Rayleigh-Ritz method a solution is obtained and compared with other models.

INTRODUCTION

Modeling of gear tooth by cantilever is a common practice [1,2]. Usually, beam and thin plate theories are used [3-5]. Because of the nature of boundary conditions of the cantilever, and for the analysis of 'thick' teeth where shear effect can not be neglected, one should use either three-dimensional approach or thick plate theory [6-8]. In this study the gear tooth is modeled as a 'thick' plate. The analysis of thick plate is based on Reissner and Mindlin plate theories [9,10]. Recently Bergan and Wang [11] have adopted a new approach for these theories. Their approach, which is used in this study, is characterized by the use of one independent variable : namely the transverse deflection of the mid-plane of the plate

Applying Rayleigh-Ritz method, a solution is obtained and compared with the results of beam modeling where shear effect is taken into consideration. A comparison is also made with the moment image and classical thin plate solutions [4,5].

* Ministry of Higher Education, Cairo, Egypt.

** The Ohio State University, Columbus, Ohio.

The paper starts with a review of the energy expressions of classical plate bending theory and of Bergan-Wang approach. This is followed by a description of Rayleigh-Ritz method used to analyze the cantilever plate. Then a square cantilever plate under uniform and concentrated loads with different thickness is analyzed. Numerical results show that beam modeling is stiffer than plate modeling and the thinner the structure the greater the difference between them. As a practical example gear teeth deflections were calculated showing that shear inclusion leads to a more flexible result .

ENERGY EXPRESSIONS

a. Thin Plate Strain Energy:

The classical thin plate theory based on Kirchoff-Love assumptions [8] gives a strain energy expression which is a function of the only transverse deflection $w(x,y)$, where x and y denote the cartesian coordinates in the mid-plane of the plate. More precisely, we have for an isotropic material the following expression for the strain energy:

$$U = \frac{1}{2} \iint K^T \; D \; K \; dxdy,$$

where

$$K^T = \left[w_{,xx} \quad w_{,yy} \quad 2w_{,xy} \right],$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

and E is the Young's modulus, h is the plate thickness, and ν is the Poisson's ratio.

Because the classical theory neglects the shear effect, its use is generally limited to thin plates where the ratio of thickness to the smaller span does not exceed 0.05 [6,7,12]. When the shear is to be considered, we usually use either Reissner [9] or Mindlin plate theory [10]. In both theories the strain energy of the deformed plate is expressed by three a priori unknown functions: the transverse deflection w(x,y), and the two transverse shear strains $\gamma_{xz}(x,y)$ and $\gamma_{yz}(x,y)$ [7,1].

Recently, Bergan and Wang [11] have modified the theory reaching to a formulation which depends only on the transverse deflection w(x,y). Their approach will be used in this study.

b. Bergan-Wang Expression for Plate Strain Energy

Assuming that the transverse shear strains : $\gamma_{xz}(x,y)$ and $\gamma_{yz}(x,y)$ are linear functions of x and y, Bergan and Wang arrived at the following energy expression:

$$U = \frac{1}{2}\iint K_b^T \; D_b \; K_b \; dx\,dy + \frac{1}{2}\iint \Gamma^T D_s \Gamma \; dx\,dy$$

Where

$$K_b = \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix} + \frac{h^2}{5(1-\nu)} \begin{bmatrix} w_{,xxxx} + w_{,xxyy} \\ w_{,yyyy} + w_{,xxyy} \\ 2(w_{,xxxy} + w_{,xyyy}) \end{bmatrix}$$

With $D_b = D$ as in classical bending theory,

and

$$\Gamma^T = \frac{h^2}{5(1-\nu)} \begin{bmatrix} w_{,xxx} + w_{,xyy} & w_{,yyy} + w_{,xxy} \end{bmatrix}$$

With $D_s = \dfrac{5\,Eh}{12(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

One can easily notice that while this approach uses only the transverse deflection w, it involves its derivatives up to the order four. It is also clear that as the thickness of the plate decreases the strain energy would be given by the classical thin theory without any numerical difficulties.

RAYLEIGH- RITZ METHOD

Consider a rectangular plate that is clamped along one of its sides (see Figure 1). Following [4,5,13] , the deflection w(x,y) is expressed as a series of products of functions of the form:

$$w(x,y) = \sum_m \sum_n A_{mn} \; X_m(x) \; Y_n(y)$$

Where the functions X_m and Y_n are the characteristic functions of uniform thickness beams in the x and y directions respectively. These functions have the appropriate beam boundary conditions [14]. In our case, they are given by:

$$X_1(x) = 1,$$

$$X_2(x) = \sqrt{3}\,(1-2\,\frac{x}{a}),$$

$$X_m(x) = \cosh a_m x + \cos a_m x - c_m(\sinh a_m x + \sin a_m x),$$

and

$$Y_n(y) = \cosh b_n y - \cos b_n y - d_n(\sinh b_n y - \sin b_n y),$$

Where a_m, b_n, c_m and d_n are tabulated terms.

and A_{mn} are the unknown coefficients.

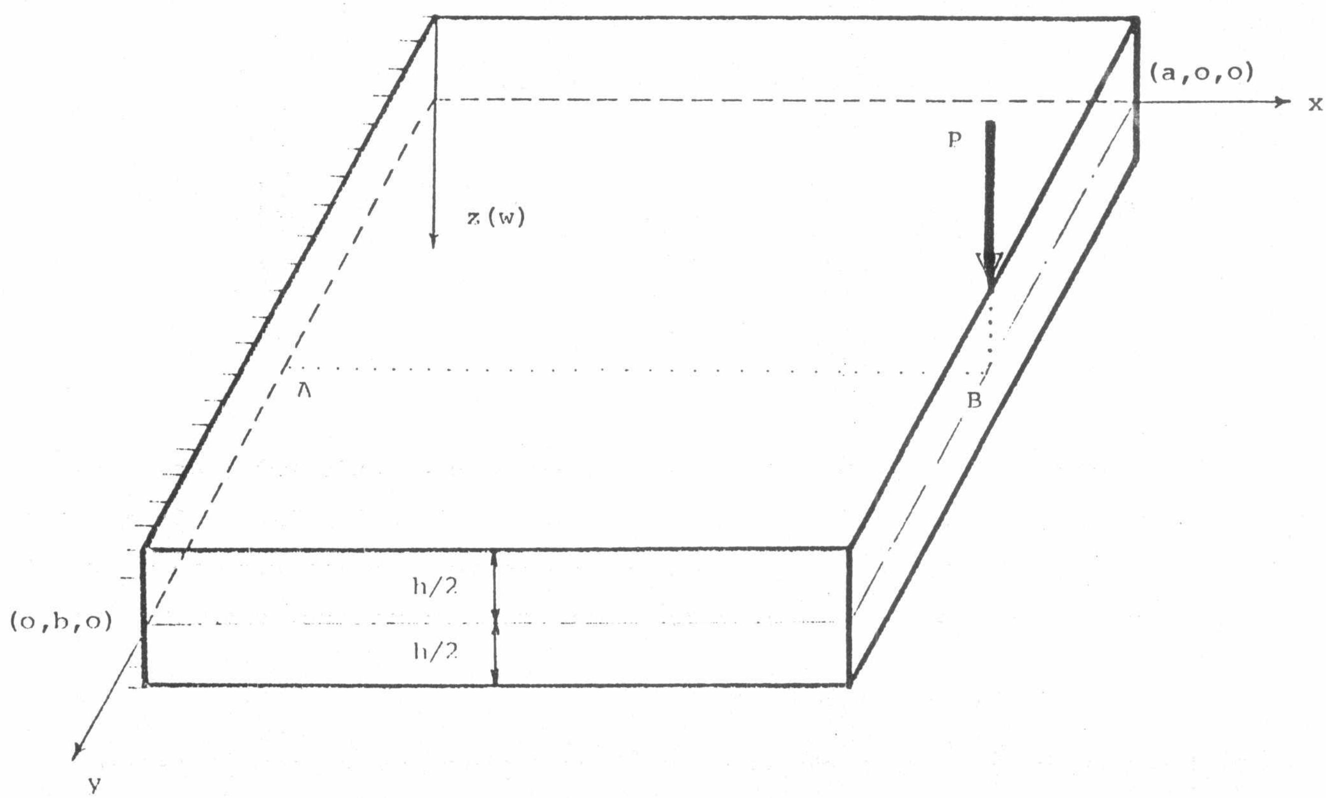

Fig. 1. Cantilever plate under concentrated end load

Application of Rayleigh-Ritz method would require a choice of a number of characteristic functions, and substitution of the expression of w into the total potential energy expression.

Then differentiating the expression with respect to the unknown coefficients A_{mn}, and equating it with zero will lead to a system of equations to be solved for these unknowns. The number of used functions was determined according to the experience gained in a previous work [4,5], in addition to the pre-implementation and re-evaluation made for this study. A subsequent paragraph shows the influence of the number of characteristic function on the obtained solution.

NUMERICAL APPLICATIONS

Example 1:

A square plate subject to a transverse uniformly distributed load q, and to a concentrated force P at point B (see Figure 1) is analyzed with different thickness to side ratio. The obtained deflection along the center line AB is shown in Figures 2 and 3.

In Tables 1 and 2 the deflection at the point B is compared with that obtained by beam analysis which consider the shear effect [15,16].

All numerical results are done with seven functions $X_m(x)$ and five functions $Y_n(y)$ [4,17]. The model problem was chosen with side length a = 0.4 inch, and Poisson's ratio $\nu = 0.3$, and Young's modulus $E = 30\times10^6$ psi.

Table 1. Cantilever Square Plate Under Uniform Load

$$w = \alpha \frac{Pa^2}{D}$$

h/a	α (Present Analysis) ($P = q\ a^2$)	α (Beam Theory) ($P = q^*\ a$)
0.1	0.688	0.139
0.3	0.528	0.148
0.5	0.262	0.167

Table 2. Cantilever Square Plate Under Concentrated Load Acting at Point B.

$$w = \alpha \frac{Pa^2}{D}$$

h/a	α (Present Analysis)	α (Beam Theory)
0.1	1.967	0.369
0.3	1.425	0.388
0.5	0.695	0.426

Effect of Number of Characteristic Functions

In addition to the experience gained by different authors [5,13,17] , it is useful to show some results which clarify and justify the chosen numbers in this study. As shown in Tables 3 and 4 the increase of number of characteristic functions leads to a more flexible model.

Table 3. Deflection Coefficient α for h/a = 0.1

No. of Characteristic Functions		Concentrated Load	Uniform Load
m	n		
3	1	1.651	0.642
5	3	1.904	0.663
7	5	1.967	0.688

Table 4. Deflection Coefficient α

for h/a = 0.5

No. of Characteristic Functions m	n	Concentrated Load	Uniform Load
3	1	0.516	0.2
5	3	0.657	0.248
7	5	0.695	0.262

Example 2. [18]

In Table 5, a comparison between the results obtained by the moment image method [1,19], and the classical plate theory [4,5], and the Bergan-Wang approach is given. As expected the Bergan-Wang approach gives a higher deflection.

Table 5. Maximum Deflection By Different Modeling

Gear Type	w (Moment Image) / w (Classical)	w (Present Study) / w (Classical)
FALK GEAR	1.033	1.138
EATON GEAR	0.790	1.164
HELICOPTER GEAR	0.809	1.247

It is useful to mention that closed form solution of Reissner [9,15] can not be applied for the solved examples where the span to chord ratio is near or equal to one. Moreover , the solution of References [19,20] neglects the effect of the free opposite sides of the cantilever.

It is also worthnoting that the concentrated load in this study is treated as a ' point ' load and not replaced by 'conical' loading as done in Reference 3.

CONCENTRATED LOAD

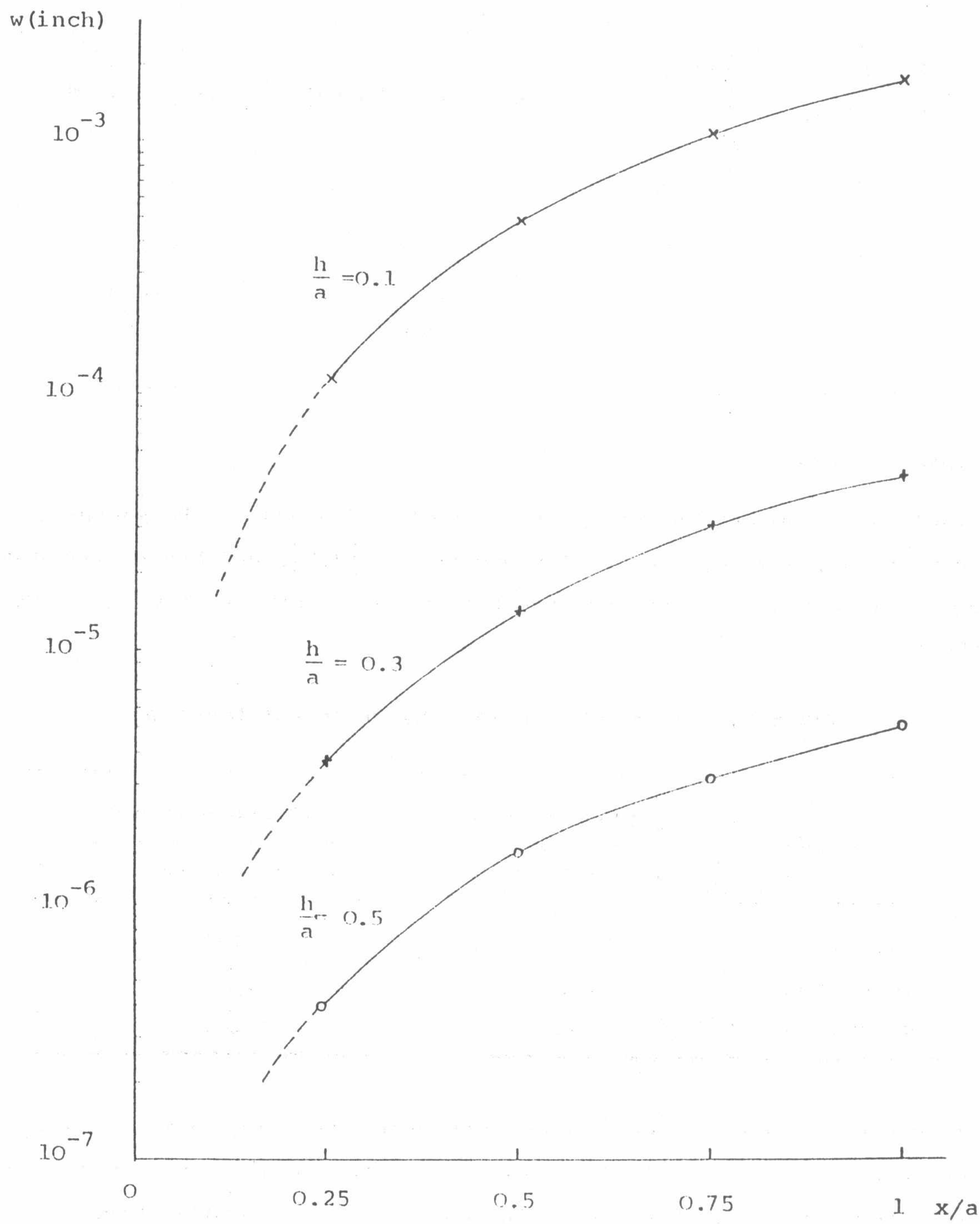

Fig. 2 Deflection w along the center line AB due to concentrated load P=1 lb at point B. (see Fig.1)

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UNIFORM LOAD

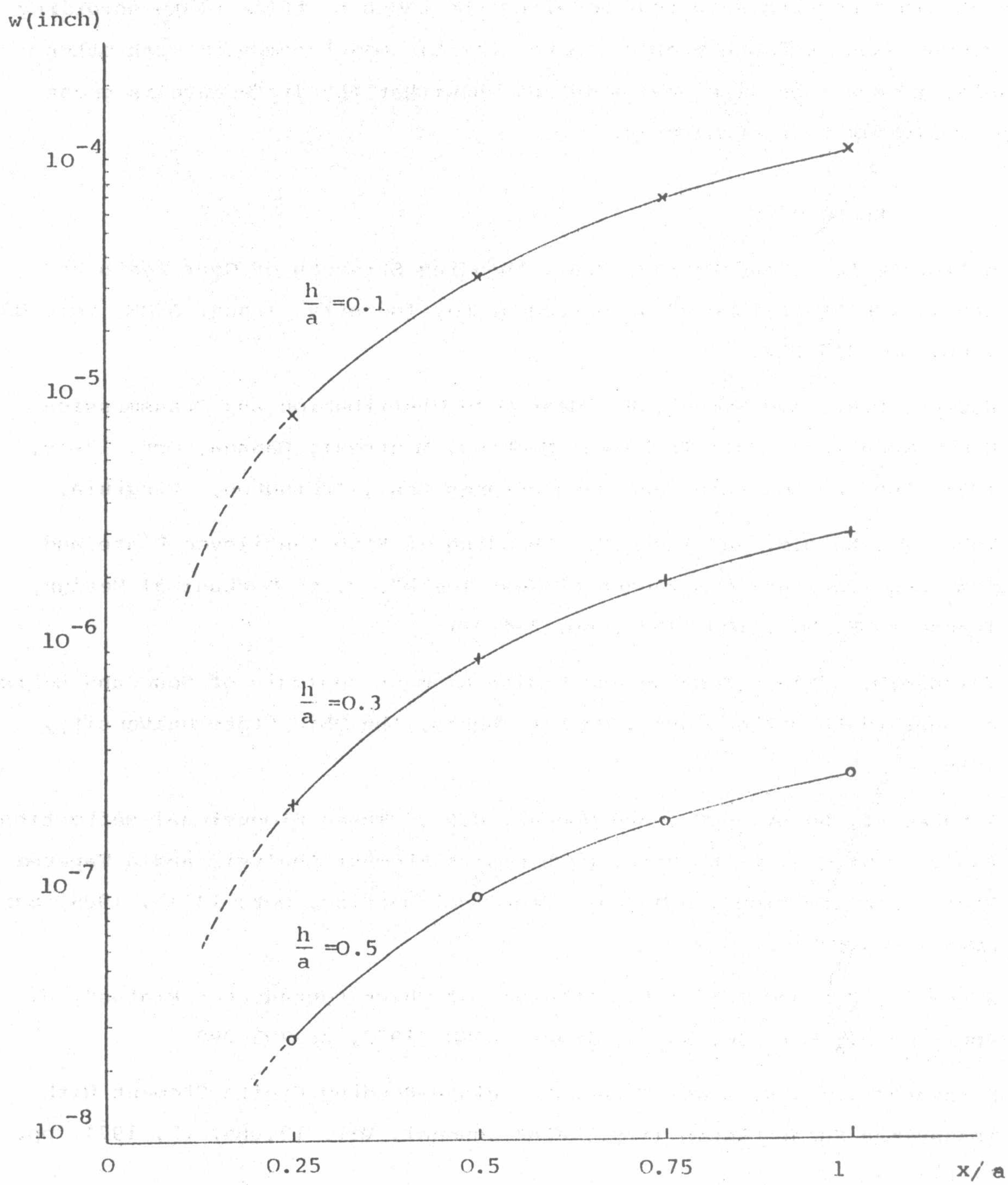

Fig.3 Deflection w along the center line AB due to uniform load q=1 psi (see Fig. 1).

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CONCLUSION

The obtained results show that modeling the tooth by thick plate according to Bergan-Wang approach yields a more flexible model compared with other models. Comparison with beam modeling shows that the difference is great especially for thin structures.

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