On Generalized Linear Exponential Distribution: Different Methods of Estimation

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Abstract

This paper concerns with various techniques for estimations from the generalized linear exponential distribution (GLED) that can be used for modeling bathtub, increasing and decreasing hazard rate (HR) behavior and was first proposed by [3]. This distribution is important since it contains as special sub-models some widely well-known distributions such as the exponential distribution (ED), the Rayleigh distribution (RD), the linear exponential distribution (LED), and the Weibull distribution (WD). The various techniques for estimations can be considered as maximum likelihood estimation (MLE), least-square estimation (LSE), weighted least square estimation (WLSE), Cramér Von-Mises estimation (CVME), and Anderson Darling estimation (ADE). These methods of estimations are used to estimate the unknown parameters of the well-known GLED. Two applications are used to show that the GLED is a viable distribution in modeling lifetime data and to compare the varying methods of estimations based on the Kolmogorov-Simnorov test with the corresponding P-value to show the optimal method. Finally, a simulation study is presented to compare the varying methods of estimation based on the mean square error (MSE) and the average absolute bias (AAB).

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1. Introduction

In studying lifetime data, One can show that the LED, which has only increasing HR, is well-known for modeling lifetime data in reliability analysis. Many authors introduced a generalization of LED to make it a model phenomenon with decreasing, unimodal and bathtub HRF. [1] studied the exponentiated linear exponential distribution (ELED) and showed that the HRF can be increasing, decreasing and bathtub shaped. The transmuted linear exponential distribution with increasing, decreasing, unimodal and bathtub HR shapes were introduced by [2].

Another generalization of LED was known as the generalized LED (GLED) and was proposed first by [3]. Several authors have considered the generalization for the GLED [[4], [5] and [6]]. Recently, [7] provided some notes on GLED in [2] and [8] proposed a new transformation called inverted GLED. The fundamental point of this paper is to study how the various estimators of the unknown parameters of a GLED behave for various sample sizes and different parameter values.

The remainder of the article is organized into six sections. Section 2 introduces some statistical functions of GLED. The MLE of GLED is studied in Section 3. Section 4 studies least square estimator and weighted least square estimator for estimating the parameters of GLED. Analysis of the Cramér Von-Mises estimator using GLED is given in Section 5. Section 6 describes the Anderson Darling estimator. Analysis of two real data sets is presented in Section 7. Section 8 presents a

simulation study to compare the different methods of estimation.

2. Generalized Linear Exponential Distribution

For a non-negative random variable X, the cumulative distribution function (CDF) and the probability density function (PDF) of GLED are given respectively by

$$F(x; c, b, \xi) = 1 - e^{-(c\,x + \frac{b}{2}x^2)^\xi}, \quad c \geq 0, \ b \geq 0, \ \xi > 0, x > 0. \quad (1)$$

and

$$f(x; c, b, \xi) = \xi\, e^{-(c\,x + \frac{b}{2}x^2)^\xi} \, (c\,x + \frac{b}{2}x^2)^{\xi-1} \, (c + b\,x) \quad (2)$$

The survival and hazard rate functions are respectively by

$$S(t; \Phi) = e^{-(c\,t + \frac{b}{2}t^2)^\xi}, \quad (4)$$

and

$$h(t; \Phi) = \xi\, (c\,t + \frac{b}{2}t^2)^{\xi-1} \, (c + b\,t) \quad , \quad t > 0, \quad (5)$$

where $\Phi = (c, b, \xi)$ is the parameter vector.

Remark 1.

For $\xi = 1$, Equation (2) reduces to

$$F(x; c, b, \xi) = 1 - e^{-(c\,x + \frac{b}{2}x^2)}, x > 0, \quad (6)$$

which is the CDF of the LED [9]. If $b = 0$ and $c = 0$ with $\xi = 1$, the ED and the RD are given respectively.

It is observed in [3] that the three-parameter GLED can have increasing, decreasing and bathtub failure rates.

3. Maximum Likelihood Estimation

MLE is likely the most broadly utilized technique for estimation in statistics. Suppose that $x_1,\ldots,x_r$ be an independent random sample of size r from GLED. From Equation (2), the log-likelihood function can be obtained as

$$\ell(\Phi) = r\log(\xi) - \sum_{i=1}^{r}(\frac{b}{2}\,x_i^2 + cx_i)^{\xi} + (\xi-1)$$
$$\sum_{i=1}^{r}\log\left(\frac{b}{2}x_i^2 + c\,x_i\right) + \sum_{i=1}^{r}\log(b\,x_i + c)\,. \qquad (7)$$

By taking the first derivative ($\ell_\Phi(\Phi) = \frac{\partial\ell}{\partial\Phi}$) of (7) w.r.t. c, b, and ξ we get

$$\ell_c(\Phi) = -\sum_{i=1}^{r}\ \xi\ x_i\left(\frac{1}{2}b\ x_i^2 + c\ x_i\right)^{\xi-1} + \sum_{i=1}^{r}\frac{(\xi-1)x_i}{\frac{1}{2}\ b\ x_i^2 + c\ x_i}$$
$$+ \sum_{i=1}^{r}\frac{1}{b\ x_i + c}\,, \qquad (8)$$

$$\ell_b(\Phi) = -\sum_{i=1}^{r}\frac{1}{2}\ \xi\ x_i^2\left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^{\xi-1} + \sum_{i=1}^{r}\frac{(\xi-1)\ x_i^2}{2\left(\frac{1}{2}\ bx_i^2 + cx_i\right)}$$
$$+ \sum_{i=1}^{r}\frac{x_i}{b\ x_i + c}, \qquad (9)$$

and

$$\ell_\xi(\Phi) = \frac{r}{\xi} - \sum_{i=1}^{r}\left(\frac{1}{2}\ bx_i^2 + c\ x_i\right)^{\xi}\log\left(\frac{1}{2}\ b\ x_i^2 + cx_i\right)$$
$$+ \sum_{i=1}^{r}\log\left(\frac{1}{2}\ bx_i^2 + c\ x_i\right), \qquad (10)$$

3.1. The parameters c, and b are known

The normal equation $\ell_\xi(\Phi) = 0$ can be written as

$$\frac{1}{\xi} = \frac{\sum_{i=1}^{r}\left(\frac{b}{2}\ x_i^2 + cx_i\right)^{\xi}\log\left(\frac{b}{2}\ x_i^2 + c\ x_i\right) - \sum_{i=1}^{r}\log\left(\frac{b}{2}\ x_i^2 + c\ x_i\right)}{r}. \qquad (11)$$

It is clear that the first derivative of the right-side hand ($\Psi_1(\xi)$) of (11) w.r.t. ξ is always positive. This means that the $\Psi_1(\xi)$ is an increasing function. Then by the graphical method [10], the MLE of ξ exists and unique.

3.2 The parameters c, b, and ξ are unknown

The MLE $\hat{\Phi}$ of Φ is given by solving the three normal equations $\ell_c(\Phi) = 0$, $\ell_b(\Phi) = 0$, and $\ell_\xi(\Phi) = 0$. These non-linear equations cannot be solved analytically see [1]. So, some numerical technique can be employed to solve these three non-linear equations.

3.3. Fisher information matrix

Since the calculation of Fisher information matrix (given by taking the expectation of the second derivative of (7)) is extremely difficult, so, it appears to be proper to approximate these expected values by their MLEs. Then, the asymptotic variance-covariance matrix is given as [see, [11]];

$$\begin{pmatrix} Var(\hat{c}) & Cov(\hat{c},\hat{b}) & Cov(\hat{c},\hat{\xi}) \\ Cov(\hat{b},\hat{c}) & Var(\hat{b}) & Cov(\hat{b},\hat{\xi}) \\ Cov(\hat{\xi},\hat{c}) & Cov(\hat{\xi},\hat{b}) & Var(\hat{\xi}) \end{pmatrix} =$$
$$\begin{pmatrix} -\ell_{cc}(\Phi) & -\ell_{cb}(\Phi) & -\ell_{c\xi}(\Phi) \\ -\ell_{bc}(\Phi) & -\ell_{bb}(\Phi) & -\ell_{b\xi}(\Phi) \\ -\ell_{\xi c}(\Phi) & -\ell_{\xi b}(\Phi) & -\ell_{\xi\xi}(\Phi) \end{pmatrix}^{-1}_{(\hat{c},\hat{b},\hat{\xi})}$$

where $\ell_{\Phi_i\Phi_j}(\Phi) = \frac{\partial^2\ell}{\partial\Phi_i\Phi_j}$, $i,j = 1,2,3$, see Appendix A. Accordingly, the ACIs based on the asymptotic variance-covariance matrix for the parameters c, b, and ξ are respectively given as:

$$\hat{c} \pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{c})}, \hat{b} \pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{b})}, \text{ and } \hat{\xi} \pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\xi})},$$

where $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

4. Least square and weighted least square estimations

The study by [12] introduces the LSE and the WLSE for evaluating the estimation of the parameters for beta distribution. These techniques will perform to estimate the parameters of GLED. For this reason, take x_i, $i = 1,\ldots,r$ as the ordered sample of a random sample of size r. At that point, the LSE of the parameters of GLED can be obtained by minimizing the following function

$$L1 = \sum_{i=1}^{r}(F(x_i;c,b,\xi) - \frac{i}{r+1})^2$$

w.r.t. the unknown parameters c, b, and ξ or by solving the following non-linear equations

$$\frac{\partial L1}{\partial c} = \sum_{i=1}^{r}2\ (1 - e^{-(c\ x_i + \frac{b}{2}x_i^2)^{\xi}} - \frac{i}{r+1})\ \delta_1(x_i;\Phi) = 0,$$
$$\frac{\partial L1}{\partial b} = \sum_{i=1}^{r}2\ (1 - e^{-(c\ x_i + \frac{b}{2}x_i^2)^{\xi}} - \frac{i}{r+1})\ \delta_2(x_i;\Phi) = 0,$$

and

$$\frac{\partial L1}{\partial \xi} = \sum_{i=1}^{r}2\ (1 - e^{-(c\ x_i + \frac{b}{2}x_i^2)^{\xi}} - \frac{i}{r+1})\ \delta_3(x_i;\Phi) = 0,$$

where

$$\delta_1(x_i;\Phi) = \xi\sum_{i=1}^{r}\ x_i\ e^{-(c\ x_i + \frac{b}{2}x_i^2)^{\xi}}(c\ x_i + \frac{b}{2}x_i^2)^{\xi-1} \qquad (12)$$

$$\delta_2(x_i;\Phi) = \xi\sum_{i=1}^{r}\frac{1}{2}\ x_i^2 e^{-(c\ x_i + \frac{b}{2}x_i^2)^{\xi}}\ (c\ x_i + \frac{b}{2}x_i^2)^{\xi-1}, \qquad (13)$$

and

$$\delta_3(x_i;\Phi) = \sum_{i=1}^{r}e^{-(c\ x_i + \frac{b}{2}x_i^2)^{\xi}}\ (c\ x_i + \frac{b}{2}x_i^2)^{\xi}\ \log[\ (c\ x_i + \frac{b}{2}x_i^2)]\,, \qquad (14)$$

To determine the WLSE of the unknown parameters of GLED, minimizes the following function

$$L2 = \sum_{i=1}^{r}\frac{(r+2)(r+1)^2}{(r-i+1)}\ (F(x_i) - \frac{i}{r+1})^2$$

w.r.t. the unknown parameters Φ or solve the non-linear equations

$$\frac{\partial L2}{\partial c} = \sum_{i=1}^{r} 2 \; \frac{(r+2)(r+1)^2}{(r-i+1)} \; (1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi} - \frac{i}{r+1}) \; \delta_1(x_i; \Phi) = 0,$$

$$\frac{\partial L2}{\partial b} = \sum_{i=1}^{r} 2 \; \frac{(r+2)(r+1)^2}{(r-i+1)} \; (1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi} - \frac{i}{r+1}) \; \delta_2(x_i; \Phi) = 0$$

and

$$\frac{\partial L2}{\partial \xi} = \sum_{i=1}^{r} 2 \; \frac{(r+2)(r+1)^2}{(r-i+1)} \; (1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi} - \frac{i}{r+1}) \; \delta_3(x_i; \Phi) = 0,$$

where $\delta_1(x_i; \Phi)$, $\delta_2(x_i; \Phi)$ and $\delta_3(x_i; \Phi)$ are given by (12), (13), and (14) respectively.

5. Cramér Von-Mises estimation

In a study conducted by [13], it was demonstrated that the bias of CVME is smaller than the other minimum distance estimator. The CVME of GLED can be given by minimizing the following function

$$W^2 = \frac{1}{12\,r} + \sum_{i=1}^{r} (F(x_i) - \frac{2\,i-1}{2\,r})^2$$

w.r.t. the unknown parameters Φ or by solving the following non-linear equations

$$\frac{\partial W^2}{\partial c} = \sum_{i=1}^{r} 2 \; (1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi} - \frac{2\,i-1}{2\,r}) \; \delta_1(x_i; \Phi) = 0,$$

$$\frac{\partial W^2}{\partial b} = \sum_{i=1}^{r} 2 \; (1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi} - \frac{2\,i-1}{2\,r}) \; \delta_2(x_i; \Phi) = 0,$$

and

$$\frac{\partial W^2}{\partial \xi} = \sum_{i=1}^{r} 2 \; (1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi} - \frac{2\,i-1}{2\,r}) \; \delta_3(x_i; \Phi) = 0,$$

where $\delta_1(x_i; \Phi)$, $\delta_2(x_i; \Phi)$ and $\delta_3(x_i; \Phi)$ are given by (12), (13), and (14) respectively.

6. Anderson Darling estimation

In another study, [14] studied the properties of ADE. Under the results gave by him, the ADE of GLED can be given by minimizing the following function

$$A^2 = -r - \sum_{i=1}^{r} \frac{(2\,i-1)}{r} (\log(\,F(x_i)\,) + \log(\,1 - F(x_{r-i+1})\,)) \,,$$

w.r.t. the unknown parameters Φ or by solving the following non-linear equations:

$$\frac{\partial A^2}{\partial c} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r} \left(\frac{\delta_1(x_i; \Phi)}{1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi}} - \frac{\delta_1(x_{r-i+1}; \Phi)}{e^{-(c\, x_{r-i+1} + \frac{b}{2}x_{r-i+1}^2)^\xi}} \right) = 0,$$

$$\frac{\partial A^2}{\partial b} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r} \left(\frac{\delta_2(x_i; \Phi)}{1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi}} - \frac{\delta_2(x_{r-i+1}; \Phi)}{e^{-(c\, x_{r-i+1} + \frac{b}{2}x_{r-i+1}^2)^\xi}} \right) = 0,$$

and

$$\frac{\partial A^2}{\partial \xi} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r} \left(\frac{\delta_3(x_i; \Phi)}{1 - e^{-(c\, x_i + \frac{b}{2}x_i^2)^\xi}} - \frac{\delta_3(x_{r-i+1}; \Phi)}{e^{-(c\, x_{r-i+1} + \frac{b}{2}x_{r-i+1}^2)^\xi}} \right) = 0,$$

where $\delta_1(x_i; \Phi)$, $\delta_2(x_i; \Phi)$ and $\delta_3(x_i; \Phi)$ are given by (12), (13), and (14) respectively

7 . Real Data Analysis

In this section, two real data sets are presented for interpretative study. For identifying the shapes of hazard rate for given data sets, the scaled TTT transform plot is given as

$$\phi_r(\tfrac{n}{r}) = \frac{\sum_{i=1}^{r} x_{i:r} + (r-n)\, x_{n:r}}{\sum_{i=1}^{r} x_i},$$

where $n = 1, \ldots, r$ and $x_{i:r}$ is the order statistics of the data [see [15]]. $K - S$ test is used for non-parametric test statistic. This test is defined as:

$$K - S = \max(|\tfrac{i}{r} - F(x_i)|, |F(x_i) - \tfrac{i-1}{r}|) \,.$$

For every data set, we compare the different methods of estimation for GLED using the K-S test with the corresponding P-value. All computations were introduced by *Wolfram Mathematica11*.

7.1. Lifetimes of 50 devices

Consider the lifetime data of 50 devices which were provided by [15]. The data set are
$\{0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32,$
$36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79,$
$82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86\}$. All estimates of the unknown parameters based on the different methods, which considered in this paper, and the values of K-S with the value of the corresponding p-value are summarized in (**Table 1**) for given data. From (**Table 1**), one can show that the MLE is considered as a good method for given data at a level of significance $\alpha = 0.05$. Also, all methods can be considered as good methods for given data at $\alpha = 0.01$. It is clear from Table 1 that the MLE is the optimal method under the K-S test with the corresponding p-value. The data are known to have a bathtub-shaped failure rate as shown by the scaled TTT-Transform plot, which has a convex shape followed by a concave shape, See **Figure (3a)**. The graphical method of ξ and $\Psi_1(\xi)$ based on Equation (11) can be shown in (**Figure 1**) for given data.

Table 1: The estimates of unknown parameters under varying methods and the $K - S$ test with the corresponding P-value for the lifetime data of 50 devices data

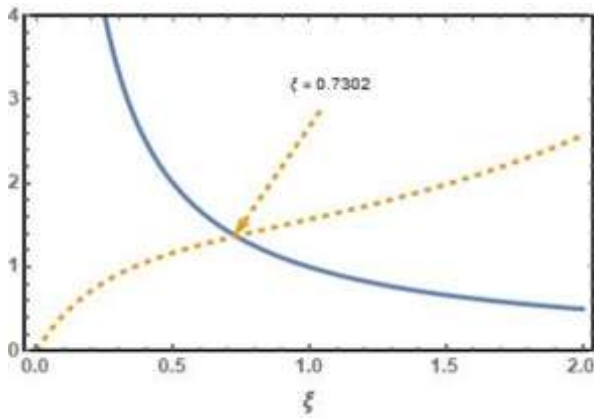| Methods	$\hat{c}$	$\hat{b}$	$\hat{\xi}$	$K - S$	$P - value$
MLE	0.0096	0.0004	0.7302	0.1799	0.0786
LSE	0.0072	0.0004	0.6457	0.2109	0.0234
WLSE	0.0096	0.0003	1.1484	0.1954	0.0439
CVME	0.0075	0.0036	0.6738	0.2040	0.0312
ADE	0.0068	0.0004	0.5776	0.2055	0.0293

Figure 1: Plot of the $\frac{1}{\xi}$ and $\Psi_1(\xi)$ functions for the lifetime data of 50 devices data
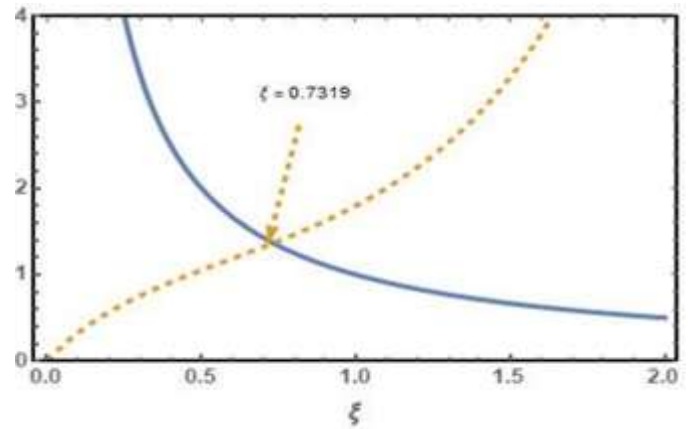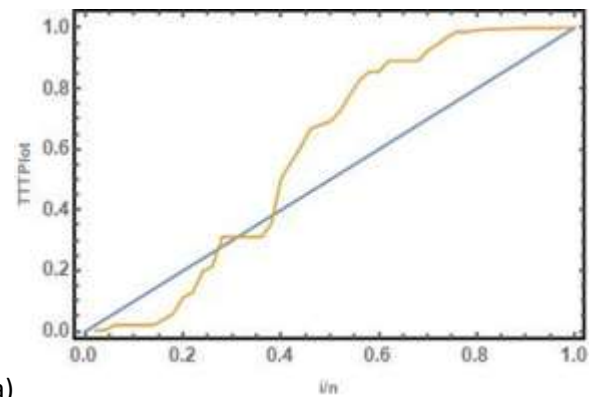

Figure 2: Plot of the $\frac{1}{\xi}$ and $\Psi_1(\xi)$ functions for the electrical appliances failure time data.
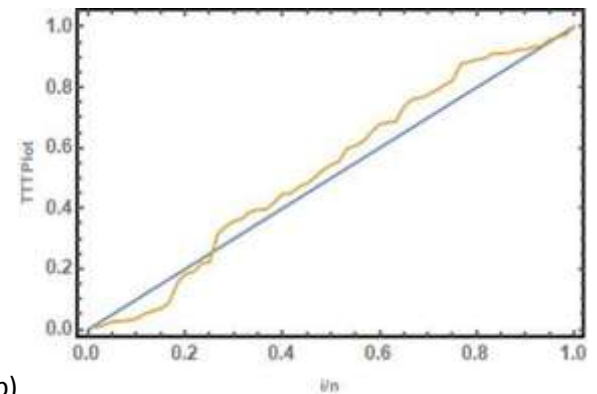
7.2. The electrical appliances failure time data

Consider the following data set from [16] consisting of the number of cycles to failure for a group of 60 electrical appliances in a life test. The failure times divided by 1000 and detailed as
$\{0.014, 0.034, 0.059, 0.061, 0.069, 0.08, 0.123, 0.142, 0.165,$
$0.21, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969,$
$0.991, 1.064, 1.088, 1.091, 1.174, 1.27, 1.275, 1.355, 1.397,$
$1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292,$
$2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248,$
$3.715, 3.79, 3.857, 3.912, 4.1, 4.106, 4.116, 4.315, 4.51, 4.584,$
$5.267, 4.299, 5.583, 6.065, 9.701\}$. All estimates of the unknown parameters based on the different methods, which considered in this paper, and the values of K-S with the value of the corresponding p-value are summarized in (**Table 2**) for given data. From (**Table 2**), one can show that all methods can be considered as good methods for given data at a level of significance $\alpha = 0.05$. It is clear from Table 2 that the ADE is the optimal method under the K-S test with the corresponding p-value. Furthermore, the TTT plot of the observed data shows that the hazard rate of the electrical appliances failure time data is bathtub which is first convex and then concave as shown in **Figure (3b)**. The graphical method of ξ and $\Psi_1(\xi)$ based on Equation (11) can be shown in (**Figure 2**) for given data.

Table 2: The estimates of unknown parameters under varying methods and the $K - S$ test with the corresponding P-value for the electrical appliances failure time data

Methods	$\hat{c}$	$\hat{b}$	$\hat{\xi}$	$K - S$	$p - value$
MLE	0.2185	0.1763	0.7319	0.0632	0.9702
LSE	0.1591	0.2029	0.6574	0.0692	0.9361
WLSE	0.1989	0.1789	0.7462	0.0723	0.9123
CVME	0.1652	0.2012	0.6787	0.0626	0.9728
ADE	0.1692	0.2073	0.6685	0.0604	0.9809

a)

b)

Figure 3: **(a)** Scaled TTT transform of the Lifetimes of 50 devices data.
(b) Scaled TTT transform of the electrical appliances failure time data.

8. Simulation Study

In this section, the simulation study is presented. The MSE and the AAB are evaluated to study the performance of varying methods of estimation which are given in the previous sections. To compare the performance of varying methods for estimating the unknown parameters of GLED, the following technique is adopted as follows:
1. Set the parameters (c, b, ξ) as (0.5,0.5,2).

2. Generate random samples from GLED with size $n = (20, 50, 80, 120)$.

3. Evaluate the varying estimates as shown in previous sections.

4. Repeat Steps 1 and 3 $N = 2000$ times.

5. Calculate the MSEs and AABs.

All numerical values of the simulation study for the different methods of estimation are reported in Tables 3 - 5. From the simulation study, one can show from Tables 3 - 5 that the:

- MSEs for all methods of estimation decrease by increasing the sample size n.
- AABs for all methods of estimation decrease by increasing the sample size n.
- Based on MSE, all methods of estimation produce smaller MSE for c compared to that of b and ξ.
- Based on AAB, all methods of estimation produce smaller AAB for c compared to that of b and ξ.

Table 3: MSEs and AABs for parameter c under proposed methods with different sample size n

n		MLE	LSE	WLSE	CVME	ADE
20	MSE	0.1310	0.1053	0.1089	0.1204	0.1117
	AAB	0.3081	0.2668	0.2721	0.2897	0.2785
50	MSE	0.1121	0.0935	0.1071	0.1012	0.0941
	AAB	0.2797	0.2441	0.2636	0.2559	0.2438
80	MSE	0.1066	0.0853	0.0981	0.0921	0.0841
	AAB	0.2657	0.2278	0.2455	0.2377	0.2306
120	MSE	0.0933	0.0758	0.0924	0.0816	0.0738
	AAB	0.2410	0.2100	0.2357	0.2183	0.2102

Table 4: MSEs and AABs for parameter b under proposed methods with different sample size n

n		MLE	LSE	WLSE	CVME	ADE
20	MSE	0.3735	0.3536	0.3286	0.4112	0.3715
	AAB	0.5013	0.4570	0.4487	0.5051	0.4824
50	MSE	0.3139	0.2837	0.3212	0.3101	0.2916
	AAB	0.4492	0.4072	0.4405	0.4311	0.4213
80	MSE	0.3006	0.2536	0.2884	0.2759	0.2516
	AAB	0.4376	0.3777	0.4093	0.3967	0.3865
120	MSE	0.2633	0.2214	0.2668	0.2411	0.2176
	AAB	0.3990	0.3481	0.3913	0.3650	0.3526

Table 5: MSEs and AABs for parameter ξ under proposed methods with different sample size n

n		MLE	LSE	WLSE	CVME	ADE
20	MSE	0.1670	0.2521	0.2077	0.1901	0.2187
	AAB	0.2952	0.3852	0.3318	0.3095	0.3548
50	MSE	0.1654	0.1827	0.1963	0.1618	0.1711
	AAB	0.3031	0.3214	0.3325	0.2908	0.3154
80	MSE	0.1625	0.1517	0.1692	0.1449	0.1453
	AAB	0.3032	0.2887	0.3008	0.2746	0.2871
120	MSE	0.1535	0.1369	0.1601	0.1354	0.1337
	AAB	0.2929	0.2721	0.2946	0.2653	0.2737

9. Conclusion

In this paper, we compared the estimation of the parameters for the three-parameter distribution IGLED using five estimation methods, namely the maximum likelihood, least-squares, weighted least-squares, Anderson Darling and Cramér Von-Mises estimations. This conclusion is also supported by the analysis of two real data sets. The K-S test with the corresponding P-value is used to show the optimal method using the two real data. The simulation study is also presented to compare the varying methods using MSEs and AABs.

Appendix A.

The second derivatives of (7) can be written as:

$$\ell_{cc}(\Phi) = -\sum_{i=1}^{r} \frac{(\xi-1)x_i^2}{\left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^2} - \sum_{i=1}^{r} \frac{1}{(b\ x_i + c)^2}$$
$$- \sum_{i=1}^{r} [\ \xi\ x_i\ \left(\xi\ x_i\left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^{\xi-2}\right.$$
$$\left. - x_i\ \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^{\xi-2}\right)]$$

$$\ell_{bb}(\Phi) = -\sum_{i=1}^{r} \frac{(\xi-1)\ x_i^4}{4\left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^2} - \sum_{i=1}^{r} \frac{x_i^2}{(b\ x_i + c)^2}$$
$$- \sum_{i=1}^{r} [\frac{1}{2}\ \xi\ x_i^2\left(\frac{1}{2}\ \xi\ x_i^2\left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^{\xi-2}\right.$$
$$\left. - \frac{1}{2}\ x_i^2\ \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^{\xi-2}\right)]$$

$$\ell_{\xi\xi}(\Phi) = -\sum_{i=1}^{r} \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)^{\xi} \log^2\left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right) - \sum_{i=1}^{r} \frac{1}{(\xi)^2}$$

$$\ell_{cb}(\Phi) = -\sum_{i=1}^{r} \frac{(\xi - 1) \ x_i^3}{2\left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)^2} + \sum_{i=1}^{r} -\frac{x_i}{(b \ x_i + c)^2}$$

$$-\sum_{i=1}^{r} [\xi \ x_i \left(\frac{1}{2} \ \xi \ x_i^2 \left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)^{\xi-2}\right.$$

$$\left.-\frac{1}{2} x_i^2 \left(\frac{1}{2} b x_i^2 + c x_i\right)^{\xi-2}\right)]$$

$$\ell_{c\xi}(\Phi) = \sum_{i=1}^{r} \frac{x_i}{\frac{1}{2} \ b \ x_i^2 + c \ x_i} - \sum_{i=1}^{r} [x_i \left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)^{\xi-1}$$

$$+ \xi \ x_i \left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)^{\xi-1} \log\left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)]$$

$$\ell_{b\xi}(\Phi) = \sum_{i=1}^{r} \frac{x_i^2}{2\left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)} - \sum_{i=1}^{r} [\frac{1}{2} \ x_i^2 \left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)^{\xi-1}$$

$$+ \frac{1}{2} \ \xi \ x_i^2 \left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)^{\xi-1} \log\left(\frac{1}{2} \ b \ x_i^2 + c \ x_i\right)]$$

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