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1 # A RIGIDITY THEOREM FOR SURFACES IN RIEMANNIAN 3-SPACES.

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4 ## ABSTRACT

Let $M : D \to V^3$ and $\bar{M} : D \to \bar{V}^3$ ($D \subset R^2$) be two isometric surfaces in the Riemannian spaces V^3 and $\bar{V}^3$ with curvatures R, $\bar{R}$ respectively.

We shall prove that the second fundamental forms of the two surfaces are the same provided that:

1- The Gaussian curvature K of M is positive.

2- M and $\bar{M}$ have the same second fundamental form on ∂D.

3- For each $d \in D$, $L_d: T_{M(d)}(V^3) \to T_{\bar{M}(d)}(\bar{V}^3)$ is the isometry determined by its restriction ℓ_d to $T_{M(d)}(M)$ which satisfies $\ell_d \circ dM = d\bar{M}$, and $L_d\left[R(x,y)z\right] = \bar{R}(L_d x, L_d y)L_d z$ for all tangent vectors $x,y,z \in T_{M(d)}(M)$

Also it is shown that the two isometric surfaces M and $\bar{M}$ satisfying the above conditions have the same Gaussian and mean curvatures at corresponding points.

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INTRODUCTION

It is known that the first fundamental forms I of two isometric surfaces are the same. This is not the case for the second fundamental forms II. However A Švec [3] studied the conditions for two infinitesimal surfaces to have the same second fundamental form. He proved that two infinitesimal isometric surfaces in E^3 have the same second fundamental form, that is the variation in the second fundamental form $\delta II = 0$ on M, provided that the Gaussian curvature $K > 0$ on the surface M, and there is a function $\lambda : M \rightarrow R$ such that the variation of the second fundamental form $\delta II = \lambda I$ on ∂M.

Our aim in this paper is to generalize Švec's theorem from the case of infinitesimal isometric surfaces in E^3 to the case of the two general isometric surfaces in Riemannian 3-spaces.

THE RIGIDITY THEOREM

Theorem: Let V^3, $\bar{V}^3$ be two Riemannian 3-spaces with curvatures R, $\bar{R}$ respectively. Let $D \subset R^2$ be a bounded domain, and let $M : D \rightarrow V^3$, $\bar{M} : D \rightarrow \bar{V}^3$ be two surfaces, such that:

i) M and $\bar{M}$ are isometric.

ii) the Gaussian curvature of M is K and $K > 0$

iii) For each $d \in D$, let L_d: $T_{M(d)}(V^3) \rightarrow T_{\bar{M}(d)}(\bar{V}^3)$ be the isometry determined by the condition that its restriction ℓ_d to $T_{M(d)}(M)$ satisfies $\ell_d \circ dM = d\bar{M}$, and $L_d\{R(x,y)z\} = \bar{R}(L_d x, L_d y)L_d z$ for each $d \in D$ and all $x,y,z \in T_{M(d)}(M)$.

iv) II and $\overline{II}$ are the second fundamental forms of M and $\bar{M}$ respectively, and $II = \overline{II}$ on the boundary ∂D.

Then $II = \overline{II}$ on D.

Proof: In the Riemannian space V^3, let $M : D \rightarrow V^3$ be a surface. For each point $m \in M$ associate an orthonormal frame $\{m, v_i\}$, i= 1,2,3. Hence there are differntial forms ω^i, ω_i^j on D such that

$$dm = \sum_{i=1}^{3} \omega^i v_i \quad , \quad dv_i = \sum_{j=1}^{3} \omega_i^j v_j , \quad \omega_i^j + \omega_j^i = 0 \quad (i,j=1,2,3), \qquad (1)$$

with the structure equations

6

$$d\omega^i = \sum_{j=1}^{3} \omega^j \wedge \omega_j^i , \quad d\omega_i^j = \sum_{k=1}^{3} \omega_i^k \wedge \omega_k^j - \frac{1}{2} \sum_{k,L=1}^{3} R_{ikL}^j \, \omega^k \wedge \omega^L , \; R_{ikL}^j + R_{iLk}^j = 0 \quad (2)$$

$(i,j,k,L=1,2,3)$.

Since dm lies in the tangent plane $T_m(M)$, hence from (1) we have

$$\omega^3 = 0. \quad (3)$$

The exterior differential of (3) gives

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0, \quad (4)$$

and hence there exist functions $a,b,c : D \to R$ such that

$$\omega_1^3 = a\omega^1 + b\omega^2 \quad , \quad \omega_2^3 = b\omega^1 + c\omega^2 . \quad (5)$$

The first and second fundamental forms of M are given respectively by

$$I = (\omega^1)^2 + (\omega^2)^2 \quad , \quad II = \omega^1 \omega_1^3 + \omega^2 \omega_2^3 = a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2 . \quad (6)$$

The Gaussian and mean curvatures of M are given respectively by

$$K = ac - b^2 \quad , \quad 2H = a + c . \quad (7)$$

Let $\bar{V}^3$ be another Riemannian space, $\bar{M} : D \to \bar{V}^3$ be another surface. For each point $\bar{m} \in \bar{M}$ associate an orthonormal frame $\{\bar{m}, \bar{v}_i\}$, i=1,2,3. Hence there are differential forms $\bar{\omega}^i, \bar{\omega}_i^j$ on D such that

$$\left.\begin{array}{l} d\bar{m} = \sum_{i=1}^{3} \bar{\omega}^i \bar{v}_i \quad , \quad d\bar{v}_i = \sum_{j=1}^{3} \bar{\omega}_i^j \bar{v}_j \quad , \quad (i,j=1,2,3) \\ d\bar{\omega}^i = \sum_{j=1}^{3} \bar{\omega}^j \wedge \bar{\omega}_j^i \; ; \quad d\bar{\omega}_i^j = \sum_{k=1}^{3} \bar{\omega}_i^k \wedge \bar{\omega}_k^j - \frac{1}{2} \sum_{k,L=1}^{3} \bar{R}_{ikL}^j \bar{\omega}^k \wedge \bar{\omega}^L , \end{array}\right\} \quad (8)$$

$\bar{R}_{ikL}^j + \bar{R}_{iLk}^j = 0, \quad (k,L = 1,2,3).$

Since $\bar{M}$ is isometric to M, then we can choose the frame $\{\bar{m}, \bar{v}_i\}$ in such a way that

$$\bar{\omega}^1 = \omega^1 \quad , \quad \bar{\omega}^2 = \omega^2 . \quad (9)$$

Let us write

$$\bar{\omega}_i^j = \omega_i^j + \tau_i^j \quad (10)$$

From (9)

. . .

$$\omega^2 \wedge \tau_1^2 = \omega^1 \wedge \tau_1^2 = 0, \tag{11}$$

hence

$$\tau_1^2 = 0 \text{ and } \bar{\omega}_1^2 = \omega_1^2 \tag{12}$$

Further we get from (2), (3), (8), (9) and (12)

$$\left.\begin{aligned} &\omega^1 \wedge \tau_1^3 + \omega^2 \wedge \tau_2^3 = 0, \\ &\omega_1^3 \wedge \tau_2^3 + \tau_1^3 \wedge \omega_2^3 + \tau_1^3 \wedge \tau_2^3 = (R_{112}^2 - \bar{R}_{112}^2)\, \omega^1 \wedge \omega^2, \\ &d\tau_1^3 = \omega_1^2 \wedge \tau_2^3 + (R_{112}^3 - \bar{R}_{112}^3)\, \omega^1 \wedge \omega^2, \\ &d\tau_2^3 = -\omega_1^2 \wedge \tau_1^3 + (R_{212}^2 - \bar{R}_{212}^3)\, \omega^1 \wedge \omega^2. \end{aligned}\right\} \tag{13}$$

Equation (13_1) implies that there exist functions $R_1, R_2, R_3 : D \rightarrow R$ such that:

$$\tau_1^3 = R_1 \omega^1 + R_2 \omega^2, \quad \tau_2^3 = R_2 \omega^1 + R_3 \omega^2. \tag{14}$$

From (14) the second fundamental form of $\bar{M}$ is then

$$\overline{II} = II + R_1 (\omega^1)^2 + 2R_2 \omega^1 \omega^2 + R_3 (\omega^2)^2. \tag{15}$$

The exterior differentiation of (12_2) gives

$$\omega_1^3 \wedge \omega_2^3 + R_{112}^2 \omega^1 \wedge \omega^2 = \bar{\omega}_1^3 \wedge \bar{\omega}_2^3 + \bar{R}_{112}^2 \omega^1 \wedge \omega^2,$$

From (5)

$$(ac - b^2) + R_{112}^2 = (\bar{a}\bar{c} - \bar{b}^2) + \bar{R}_{112}^2,$$

from (7) it follows that

$$K + R_{112}^2 = \bar{K} + \bar{R}_{112}^2. \tag{16}$$

From (5), (14), (13_2) and (16) we get

$$aR_3 - 2bR_2 + cR_1 + R_1 R_3 - R_2^2 = \bar{K} - K. \tag{17}$$

From $(13_{3,4})$ and (14)

$$\left.(dR_1 - 2R_2 \omega_1^2) \wedge \omega^1 + \left\{ dR_2 + (R_1 - R_3)\omega_1^2 \right\} \wedge \omega^2 = (R_{112}^3 - \bar{R}_{112}^3)\, \omega^1 \wedge \omega^2, \right\} \tag{18}$$

$$\left.\begin{aligned}&\left\{dR_2+(R_1-R_3)\omega_1^2\right\}\wedge\omega^1+(dR_3+2R_2\omega_1^2)\wedge\omega^2=(R^3_{212}-\bar{R}^3_{212})\,\omega^1\wedge\omega^2,\end{aligned}\right\}\quad(18)$$

and hence there exist functions $S_1,\ldots,S_4 : D \longrightarrow R$ each that

$$\left.\begin{aligned}dR_1-2R_2\omega_1^2 &= S_1\omega^1+(S_2+\bar{R}^3_{112})\omega^2,\\ dR_2+(R_1-R_3)\omega_1^2 &= (S_2+R^3_{112})\omega^1+(S_3+\bar{R}^3_{212})\omega^2,\\ dR_3+2R_2\omega_1^2 &= (S_3+R^3_{212})\omega^1+S_4\,\omega^2.\end{aligned}\right\}\quad(19)$$

From (2) and (5)

$$\left.\begin{aligned}&(da-2b\,\omega_1^2)\wedge\omega^1+\left\{db+(a-c)\omega_1^2\right\}\wedge\omega^2=-R^3_{112}\,\omega^1\wedge\omega^2,\\ &\left\{db+(a-c)\,\omega_1^2\right\}\wedge\omega^1+(dc+2b\omega_1^2)\wedge\omega^2=-R^3_{212}\,\omega^1\wedge\omega^2,\end{aligned}\right\}\quad(20)$$

and we may write

$$\left.\begin{aligned}da-2b\,\omega_1^2 &= \alpha\,\omega^1+(\beta+\tfrac{1}{2}R^3_{112})\,\omega^2,\\ db+(a-c)\,\omega_1^2 &= (\beta-\tfrac{1}{2}R^3_{112})\,\omega^1+(\gamma+\tfrac{1}{2}R^3_{212})\,\omega^2,\\ dc+2b\;\omega_1^2 &= (\gamma-\tfrac{1}{2}R^3_{212})\,\omega^1+\delta\,\omega^2.\end{aligned}\right\}\quad(21)$$

On differentiating (17) and substituting from (19) and (21) the coefficient of ω_1^2 vanishes. Hence the coefficient of each of ω^1 and ω^2 will be equal to zero, which gives

$$\left.\begin{aligned}&(c+R_3)S_1-2(b+R_2)S_2+(a+R_1)S_3=-(\gamma+\tfrac{1}{2}R^3_{212})R_1\\ &\quad+2(\beta+\tfrac{1}{2}R^3_{112})R_2-\alpha R_3-a\,R^3_{212}+2b\,R^3_{112}+(\bar{K}-K)_1,\\ &(c+R_3)S_2-2(b+R_2)S_3+(a+R_1)S_4=-\delta R_1+2(\gamma+\tfrac{1}{2}R^3_{212}+\bar{R}^3_{212})R_2\\ &\quad-(\beta+\tfrac{1}{2}R^3_{112}+\bar{R}^3_{112})R_3+2b\bar{R}^3_{212}-c\bar{R}^3_{112}+(\bar{K}-K)_2.\end{aligned}\right\}\quad(22)$$

In D, let us choose coordinates (u,v) such that

$$\omega^1=r\,du\,,\quad \omega^2=s\,dv\,,\quad r=r(u,v)\neq 0,\quad s=s(u,v)\neq 0 \qquad (23)$$

which implies that

$$\omega_1^2 = -\Delta^{-1}\frac{\partial r}{\partial v}du + r^{-1}\frac{\partial \Delta}{\partial u}dv . \qquad (24)$$

From (19) , (23) and (24) we get

$$\left.\begin{aligned}
&\frac{\partial(R_1-R_3)}{\partial u}du+\frac{\partial(R_1-R_3)}{\partial v}dv-4R_2\left(-\Delta^{-1}\frac{\partial r}{\partial v}du+r^{-1}\frac{\partial\Delta}{\partial u}dv\right)=\\
&(S_1-S_3-R^3_{212})r\,du+(S_2-S_4+\bar{R}^3_{112})\Delta\,dv ,\\
&\frac{\partial R_2}{\partial u}du+\frac{\partial R_2}{\partial v}dv+(R_1-R_3)\left(-\Delta^{-1}\frac{\partial r}{\partial v}du+r^{-1}\frac{\partial\Delta}{\partial u}dv\right)=\\
&(S_2+R^3_{112})r\,du+(S_3+\bar{R}^3_{212})\Delta\,dv.
\end{aligned}\right\} \qquad (25)$$

From (25) it follows that

$$\left.\begin{aligned}
r\Delta S_1 &= \Delta\frac{\partial(R_1-R_3)}{\partial u}+r\frac{\partial R_2}{\partial v}+\frac{\partial\Delta}{\partial u}(R_1-R_3)+4\frac{\partial r}{\partial v}R_2+r\Delta(R^3_{212}-\bar{R}^3_{212}),\\
r\Delta S_2 &= \Delta\frac{\partial R_2}{\partial u}-\frac{\partial r}{\partial v}(R_1-R_3)-r\Delta R^2_{112},\\
r\Delta S_3 &= r\frac{\partial R_2}{\partial u}+\frac{\partial\Delta}{\partial u}(R_1-R_3)-r\Delta\bar{R}^3_{212},\\
r\Delta S_4 &= -r\frac{\partial(R_1-R_3)}{\partial u}+\Delta\frac{\partial R_2}{\partial u}-\frac{\partial r}{\partial v}(R_1-R_3)+4\frac{\partial\Delta}{\partial u}R_2-r\Delta(R^3_{112}-\bar{R}^3_{112}).
\end{aligned}\right\} \qquad (26)$$

Now, let us turn our attention to condition (iii) of our theorem, for x,y,z $\in T_{M(d)}(M)$, let

$$x = x^1v_1+x^2v_2, \quad y = y^1v_1+y^2v_2 , \quad z = z^1v_1+z^2v_2 , \quad (x^3=y^3=z^3=0). \qquad (27)$$

We have

$$R(x,y)z = \sum_{l=1}^{3} R^l_{ijk}\, x^j y^k z^i v_l , \qquad (i,j,k=1,2.)$$

$$= R^2_{112}x^1y^2z^1v_2+R^3_{112}x^1y^2z^1v_3-R^2_{112}x^2y^1z^1v_2-R^3_{112}x^2y^1z^1v_3$$

$$-R^2_{112}x^1y^2z^2v_1+R^3_{212}x^1y^2z^2v_3+R^2_{112}x^2y^1z^2v_1-R^3_{212}x^2y^1z^2v_3$$

$$= (x^2y^1-x^1y^2)\left\{R^2_{112}(z^2v_1-z^1v_2)-(R^3_{112}z^1+R^3_{212}z^2)v_3\right\} \qquad (28)$$

Since $L_d v_i = \bar{v}_i$, then .

$$L_d\left\{R(x,y)z\right\} = (x^2y^1-x^1y^2)\left\{R^2_{112}(z^2\bar{v}_1-z^1\bar{v}_2)-(R^3_{112}z^1+R^3_{212}z^2)\bar{v}_3\right\},$$
$$\bar{R}(L_dx,L_dy)L_dz = (x^2y^1-x^1y^2)\left\{\bar{R}^2_{112}(z^2\bar{v}_1-z^1\bar{v}_2)-(\bar{R}^3_{112}z^1+\bar{R}^3_{212}z^2)\bar{v}_3\right\}. \qquad (29)$$

Since from the condition $L_d\left\{R(x,y)z\right\} = \bar{R}(L_dx,L_dy)L_dz$ for each $d \in D$ and all $x,y,z \in T_{M(d)}(M)$, it follows from (29) that we have on M

$$R^2_{112} = \bar{R}^2_{112}, \quad R^3_{112} = \bar{R}^3_{112}, \quad R^3_{212} = \bar{R}^3_{212}. \qquad (30)$$

Hence from (16) we get

$$K = \bar{K} \qquad (K > 0). \qquad (31)$$

Using (31), equation (17) can be written in the forms

$$(2a+2c+R_1+R_3)R_1-2(2b+R_2)R_2-(2a+R_1)(R_1-R_3) = 0,$$
or
$$(2c+R_3)(R_1-R_3)-2(2b+R_2)R_2+(2a+2c+R_1+R_3)R_3 = 0. \qquad (32)$$

From (10) and (14) we get,

$$\bar{\omega}^3_1\wedge\bar{\omega}^2+\bar{\omega}^1\wedge\bar{\omega}^3_2 = 2H\,\omega^1\wedge\omega^2+(R_1+R_3)\,\omega^1\wedge\omega^2 = 2\bar{H}\,\omega^1\wedge\omega^2, \qquad (33)$$

Hence

$$2(H+\bar{H}) = 2a+2c+R_1+R_3. \qquad (34)$$

Since $\bar{H}^2 > \bar{K} = K > 0$, $H^2 > K$, imply $2(H+\bar{H}) > 0$,

then from (32) and (34),

$$R_1 = (H+\bar{H})^{-1}\left\{(2b+R_2)R_2+\frac{1}{2}(2a+R_1)(R_1-R_3)\right\},$$
$$R_3 = (H+\bar{H})^{-1}\left\{(2b+R_2)R_2-\frac{1}{2}(2c+R_3)(R_1-R_3)\right\}. \qquad (35)$$

From (26), (35) and (22) we get

$$\left.\begin{aligned} &s(c+R_3)\frac{\partial(R_1-R_3)}{\partial u} - 2s(b+R_2)\frac{\partial R_2}{\partial u} + r(a+c+R_1+R_3)\frac{\partial R_2}{\partial v} = \\ &\qquad f_1(R_1-R_3) + f_2 R_2, \\ &-r(a+R_1)\frac{\partial(R_1-R_3)}{\partial v} + s(a+c+R_1+R_3)\frac{\partial R_2}{\partial u} - 2r(b+R_2)\frac{\partial R_2}{\partial v} = \\ &\qquad f_3(R_1-R_3) + f_4 R_2. \end{aligned}\right\} \tag{36}$$

The quadratic form $\emptyset$ of (36) is equivalent to

$$\emptyset = -(a+c+R_1+R_3)\left\{ r^2(a+R_1)\mu^2 + 2rs(b+R_2)\mu\nu + s^2(c+R_3)\nu^2 \right\}. \tag{37}$$

Let the discriminant of $\emptyset$ be $-\Delta$, then

$$\Delta = r^2 s^2 (a+c+R_1+R_3)^2 \left\{ (a+R_1)(c+R_3) - (b+R_2)^2 \right\}, \tag{38}$$

from (7), (17), (31) and $(a+c+R_1+R_3) = 2\bar{H} > 0$ equ (38) reduces to

$\Delta = r^2 s^2 K(a+c+R_1+R_3)^2 > 0$. Hence $\emptyset$ is definite and (36) is elliptic, and from (iv) we get $R_1-R_3 = R_2 = 0$ in D. From (35) we get $R_1 = R_2 = R_3 = 0$ inside D. Then from (33) we get $H = \bar{H}$, and from (15) we get $II = \overline{II}$ in D, which proves the theorem.

Q.E.D.

6

CONCLUSION

We conclude that the theorem of A. Švec [3] can be generalized from two infinitesimal isometric surfaces to the case of two general isometric surfaces in Riemannian 3-spaces. Moreover if $V^3 = \bar{V}^3 = E^3$ the condition (iii) is automatically satisfied since $R^{l}_{ijk} = \bar{R}^{l}_{ijk} = 0$.

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