1 # A NEW MEASURE FOR THE STABILITY OF HYPERSURFACES
2 # IN THE N- DIMENSIONAL EUCLIDEAN SPACE

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4 ABSTRACT

A new measure of stability on hypersurfaces in E^n is introduced. It is the integral

$$\tau(f) = \int_M H_1 H_{n-2} \, d\sigma$$

where H_i is the i^{th} mean curvature of the hypersurface M. The general conditions for the stability of M under normal deformations are found. The stability on M under II- infinitismal and III- infinitismal deformations and the stability of M under the normal deformations which admit the relation

$$\delta II = \frac{1}{2}(n-1) H_1 \delta I$$

are also discussed.

§ 1. Preface:

Let $M \subset E^n$ be an orientable C^∞ hypersurface and let $f: M \rightarrow E^n$ be a smooth immersion of M into E^n . Let

$$(1) \quad \tau(f) = \int_M H_1 H_{n-2} \, d\sigma$$

where H_1 is the first mean curvature and H_{n-2} is the $(n-2)^{th}$ mean curvature of the immersed hypersurface and $d\sigma$ is its volume element.

If $[a_{ij}]$ is the matrix of second fundamental form of M then,

$$(2) \quad H_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} a_{ii}, \quad H_2 = 1/\binom{n-1}{2} \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{vmatrix}$$

$$H_3 = 1/\binom{n-1}{3} \sum_{i<j<l} \begin{vmatrix} a_{ii} & a_{ij} & a_{il} \\ a_{ji} & a_{jj} & a_{jl} \\ a_{li} & a_{lj} & a_{ll} \end{vmatrix} \quad \ldots\ldots$$

$$H_{n-1} = \det [a_{ij}] = K ,$$

where K is the Gaussian curvature of M. Consider the conformally invariant function

$$(3) \quad \psi = (n-1)^2 \, (H_1 H_{n-2} - K)$$

when the matrix $[a_{ij}]$ is diagonalized i.e when $a_{ij}=0$ for $i \neq j$ and $a_{ii} = k_i$ the principle curvatures of M, the function takes the form

$$(4) \quad \psi = \sum_{i<j} k_1 \ldots k_{i-1} k_{i+1} \ldots k_{j-1} k_{j+1} (k_i - k_j)^2$$

For hyperovaloids and the hypersurfaces which are diffeomorphic to the hyperspheres the function $H_1 H_{n-2} - K \geq 0$ on M, we get.

$$(5) \quad \int_M H_1 H_{n-2} \, d\sigma \geq \int_M K \, d\sigma$$

Thus

$$(6) \quad \int_M H_1 H_{n-2} \, d\sigma \geq A$$

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where A is the area of the unit hypersphere, and the equality holds for the hyperspheres.

Consider the integral $\tau(f)$ in (1) and make a normal deformation f_t to hypersuface M so that $f_t(M)$ is obtained from $f(M) = f_{t=0}(M)$ by a displacement along the normal for $t \in [-\frac{1}{2}, \frac{1}{2}]$. Write the variation of f as :

(7) $$\delta f = \frac{\partial f}{\partial t}\Big|_{t=o}$$

The II-infinitismal deformation is the normal deformation which preserves the second fundamental form II of M, i.e $\delta II = o$.

The III-infinitismal deformation is that which preserves the third fundamental form III i.e $\delta III = o$.

T.J. Willmore [4] used the measure $\tau'(f) = \int_M H_1^{n-1} d\sigma$ to distinguish the closed orientable C^∞ hypersurfaces in E^n with first mean curvature H_1 which admit the infimum of $\tau'(f)$. B.Y. Chen [1] solved it as a variational problem, he used $\tau'(f) = \int_M H^{n-1} d\sigma$ as a measure of stability for the closed orientable C^∞ hypersurfaces in E^n. The integral $\int_M H_1 H_{n-2}\, d\sigma$ which I introduce and the integral $\int_M H_1^{n-1} d\sigma$ cioncide for surfaces immersed in E^3.

I shall use $\tau(f)$ in (1) to find the conditions of stability for an orientable C^∞ hypersurfaceM immersed in E^n. A hypersurface M is said to be stable if $\delta\tau = o$.

§ 2. Introduction:

Let us take a field of orthonormal moving frames $\{m, v_i, i=1,\dots,n\}$ on M such that $v_1,\dots,v_{n-1} \subset T_m(M)$. Let the dual frame be $\{\omega^1,\dots,\omega^n\}$. Let the normal deformation f at each point $m \in M$ be given by

(8) $$f(m) = m + th\, v_n$$

where $h: M \to R$, $t \in [-\frac{1}{2}, \frac{1}{2}]$

Let the connection on M be denoted by ∇ then,

(9) $\nabla m = \omega^i v_i \quad , \quad \omega^n = o$

(10) $\nabla v_i = \omega_i^j v_j + \omega_i^n v_n$

(11) $\nabla v_n = - \sum_i \omega_i^n v_i$

(12) $\nabla h = h_i \omega^i$

(13) $\nabla h_i - h_j \omega_i^j = h_{ij} \omega^j$

where h_i and h_{ij} are the first and second covariant derivatives of h respectively.

By exterior differentiation of $(9)_2$ we get :

(14) $\omega_i^n = a_{ij} \omega^j \quad , \quad a_{ij} = a_{ji} \ , \ a_{ij} : M \to R$

$[a_{ij}]$ is the matrix of the second fundamental form. Diagonalizing $[a_{ij}]$, $a_{ij} = o$ for $i \neq j$ and $a_{ii} = k_i$ the principle curvatures of M, we get :

(15) $\nabla v_i = \omega_i^j v_j + k_i \omega^i v_n$

(16) $\nabla v_n = - \sum_i k_i \omega^i v_i$

From (8) , (9) , (11) and (12) we get :

(17) $\nabla(f(m)) = f_* = \sum_i (1 - th\, k_i) \omega^i v_i + th_i \omega^i v_n$

since

$\bar{v}_i = f_* v_i = v_i(f(m)) = (\nabla(f(m)) (v_i)$

where $f_* : T_m (M) \to T_{f_t(m)} f_t(M)$, then

(18) $\bar{v}_i = (1 - th\, k_i) v_i + th_i v_n$

thus from (7) we get:

(19) $\delta v_i = h_i v_n - h k_i v_i$

From (17) , (18) :

(20) $\bar{\omega}^i = \omega^i (1 - th\, k_i)$

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(21) $\bar{v}_n = [\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}]$

where[ ]indicates the vector product of the vectors included inside the two brackets. let

$\bar{v}_i = \lambda_i v_i + \mu_i v_n$

where $\lambda_i = 1 - th\, k_i$, $\mu_i = th_i$

then

(22) $\bar{v}_n = \lambda_1 \lambda_2 \ldots \lambda_{n-1} v_n - \sum_j \lambda_1 \ldots \lambda_{j-1} \mu_j \lambda_{j+1} \ldots \lambda_{n-1} v_j$

$\bar{v}_n = \Pi_i (1-th\, k_i)\, v_n - \Sigma_j [(th_j)\, \Pi_{i\neq j}\, (1-th\, k_i)]\, v_j$

(23) $\bar{v}_n = (1 - th\, \Sigma_i\, k_i + t^2 h^2 \sum_{i\neq j} k_i k_j + \ldots)\, v_n$

$- \Sigma_j (th_j - t^2 h^2 \sum_{\substack{s \\ s\neq j}} k_j k_s + \ldots)\, v_j$

thus

$\delta v_n = - h \sum_i k_i v_n - \sum_j h_j v_j$

But $(n-1)\, H_1 = \sum_i k_i$, then

(24) $\delta v_n = -\Sigma_j\, h_j\, v_j - (n-1)\, H_1 h\, v_n$

The volume element $d\sigma$ is given by

(25) $d\sigma = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^{n-1}$

After deformation, the volume element becomes

(26) $d\bar{\sigma} = \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \ldots \wedge \bar{\omega}^{n-1}$

From (20) , (25), and (26) we get

$d\bar{\sigma} = \Pi_i\, (1-th\, k_i)\, d\sigma$

Then

(27) $d\bar{\sigma} = (1 - t(n-1)\, H_1 h + \ldots)\, d\sigma$

we deduce that the variation in volume element is given by :

(28) $\delta(d\sigma) = -(n-1)\, H_1 h\, d\sigma$

Let us write (23) in the form :

(29) $\bar{v}_n = -t \sum_j h_j\, v_j + (1-t(n-1)\, H_1 h)\, v_n + \ldots\ ,$

Then

(30) $\nabla \bar{v}_n = -\sum_j [\{1-t(n-1)\, H_1 h\}\, k_j\, \omega^j + t h_{jk}\, \omega^k]\, v_j$

$- t \sum_j (h_j k_j + (n-1)\, H_1 h_j + (n-1)\, h\, H_{1,j})\, \omega^j\, v_n$

By virtue of (16) we write

(31) $\nabla \bar{v}_n = -\sum_j \bar{k}_j\, \bar{\omega}^j\, \hat{v}_j \quad , \hat{v}_j = \bar{v}_j / \| \bar{v}_j \|$

From (18) , (20) and (31) we get

(32) $-< \nabla \bar{v}_n(\bar{v}_i), \hat{v}_i > = \bar{k}_i = -(1-thk_i) < \nabla \bar{v}_n(v_i), v_i >$

From (30) and (32) we get :

(33) $\bar{k}_i = [(1-t(n-1)\, H_1 h)\, k_i + t h_{ii}]\, (1-th\, k_i)$

$= (1-t(n-1)\, H_1 h)\, k_i + t h_{ii} - th\, k_i^2 + t^2 (\ldots) + \ldots$

Thus

(34) $\delta k_i = -(n-1)\, H_1 h\, k_i + h_{ii} - h\, k_i^2$

and

(35) $\sum k_i^2 = (n-1)^2\, H_1^2 - (n-1)(n-2)\, H_2$

From (35),(34) we get :

(36) $\delta H_1 = \frac{1}{(n-1)} [\Delta h - 2h\, H_1^2\, (n-1)^2 + (n-1)\, (n-2)\, H_2 h\ ]$

We write

$$H_{n-2} = \frac{1}{(n-1)} \sum_i \frac{K}{k_i}$$

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But Since

$$K = k_1 k_2 \ldots k_j \ldots k_{n-1}$$

$$(37) \quad \delta K = \sum_j \frac{K}{k_j} \delta k_j$$

from (34) and (37) then

$$(38) \quad \delta K = -n(n-1) H_1 h K + \sum_j \frac{K}{k_j} h_{jj}$$

$$\delta H_{n-2} = \frac{1}{n-1} \Sigma_i \left(\frac{1}{k_i} \delta K - \frac{K}{k_i^2} \delta k_i \right)$$

$$(39) \quad \delta H_{n-2} = h \left[K - (n-1)^2 H_1 H_{n-2} \right] + \frac{K}{(n-1)} \Sigma_j \frac{h_{jj}}{k_j} \left(\Sigma_i \frac{1}{k_i} - \frac{1}{k_j} \right)$$

$$(40) \quad \delta\tau = \int_M \frac{H_{n-2}}{n-1} \left[\Sigma_j h_{jj} - 2h (n-1)^2 H_1^2 + (n-1)(n-2) H_2 h \right] d\sigma +$$

$$+ \int_M H_1 \left[hK - (n-1)^2 H_1 H_{n-2} h + \frac{K}{n-1} \Sigma_j \frac{h_{jj}}{k_j} \left(\Sigma_i \frac{1}{k} - \frac{1}{k_j} \right) \right] \cdot$$

$$\cdot d\sigma - \int_M (n-1) H_1^2 H_{n-2} h \, d\sigma$$

$$(41) \quad \delta\tau = \int_M \left[\frac{1}{n-1} \left\{ H_{n-2} \Sigma_j h_{jj} + H_1 H_{n-2} \Sigma_j \frac{h_{jj}}{k_j} - (n-1) H_1 K \sum_j \frac{h_{jj}}{k_j^2} \right\} + \right.$$

$$\left. + h \left\{ -(n+2)(n-1) H_1^2 H_{n-2} + (n-2) H_2 H_{n-2} + H_1 K \right\} \right] d\sigma$$

§ 3. Main Results :

Theorem (1) Let $M \subset E^n$ be an orientable C^∞ hypersurface with unit normal vector field v_n. Let $f : m \mapsto m + th\, v_n$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, $h : M \to R$ $\forall$ $m \in M$ be a normal deformation of M. Further let:

$$(42) \frac{1}{n-1} H_{n-2} \Sigma_j h_{jj} + H_1 H_{n-2} \Sigma_j h_{jj}/k_j - H_1 K (\Sigma_j h_{jj}/k_j^2)/(n-1) + \{ H_1 K +$$

$$+ (n-2) H_2 H_{n-2} - (n+2)(n-1) H_1^2 H_{n-2} \} h = 0$$

then M is stable under any normal deformation.

Proof :
is obtained at once by equating the right hand side of (41) to zero.
Let I, II, II denote the first, the second and the third fundamental forms on M, then

$$I = \langle \nabla m, \nabla m \rangle$$
$$II = -\langle \nabla m, \nabla v_n \rangle$$
$$III = \langle \nabla v_n, \nabla v_n \rangle$$

and we get from (17)

$$\bar{I} = \sum_j (\omega^j)^2 - 2th\, k_j (\omega^j)^2 + t^2(\ \ldots)$$

Then,

$$(43)\quad \delta I = -2h\, k_j\ (\omega^j)^2$$

From (17) and (30) we get

$$(44)\quad \overline{II} = \Sigma_i\, k_i (\omega^i)^2 + t\Big[\sum_{i,j} h_{ij}\, \omega^i\, \omega^j - (n-1) H_1 h \Sigma_i k_i (\omega^i)^2 - h\, \Sigma_i (k_i \omega^i)^2\Big] + t^2(\ldots)$$

Then

$$(45)\quad \delta II = \sum_{i,j} h_{ij}\, \omega^i\, \omega^j - (n-1)\, H_1 h\, \Sigma_i\, k_i\ (\omega^i)^2 - h\, \Sigma_i (k_i \omega^i)^2$$

From (30) we get

$$(46)\quad \overline{III} = \Sigma_i (k_i \omega^i)^2 + 2t\, \Sigma_i \Big[\, k_i h_{ij} \omega^i\, \omega^j - (n-1) H_1 h (k_i \omega^i)^2\Big] + t^2(\ldots)$$

Then

$$(47)\quad \delta III = 2\, \Sigma_i\, \Big[k_i h_{ij}\, \omega^i \omega^j - (n-1) H_1 h (k_i \omega^i)^2\Big]$$

Theorem (2) Let $M \subset E^n$ be an orientable C^∞ hypersurface stisfying the relation:

$$(48)\qquad H_1^2 H_{n-2} = 0$$

Then M is stable under the II-infinitismal normal deformations.

Proof ; From (45) by putting $\delta II = 0$ then

(49) $h_{ii} = (n-1)\ H_1 h k_i + h\ k_i^2$

substituting from (49) in (42) we get

$$H_{n-2}\left[(n-1)\ H_1\ \Sigma_i k_i + \Sigma_i k_i^2\right] + H_1 H_{n-2}\left[(n-1)^2\ H_1 + \Sigma_i k_i\right](n-1) -$$

$$- H_1 K\left[(n-1)\ H_1\ \Sigma_i \frac{1}{k_i} + (n-1)\right] +$$

$$+ (n-1)\ \{H_1 K + (n-2)\ H_2 H_{n-2} - (n+2)(n-1) H_1^2\ H_{n-2}\} = 0$$

and this leads to (48)

Theorem (3) Let $M \subset E^n$ be an orientable C^∞ hypersurface satisfying the relation:

(50) $H_1 K + (n-2)\ H_2 H_{n-2} - 3(n-1)\ H_1^2\ H_{n-2} = 0$

Then M is stable under the III-infinitismal normal deformations.

Proof From (47) by putting $\delta III = 0$ then

(51) $h_{ii} = (n-1)\ H_1 k_i h$

Substituting from (51) in (42) then

(52) $(n-1)\ H_1 H_{n-2} \Sigma_i\ k_i + (n-1)^3\ H_1^2\ H_{n-2} - (n-1)\ H_1^2\ K\ \Sigma_i \frac{1}{k_i} + C = 0$

where $c = (n-1)\ \{H_1 K + (n-2)\ H_2 H_{n-2} - (n-1) H_1^2 H_{n-2} (n+2)\}$, this leads to the condition (50)

Theorem (4) Let $M \subset E^n$ be an orientable C^∞ hypersurface satisfying the relation

(53) $H_1^2 H_{n-2} = 0$

Then M is stable under the normal deformations which admit the relation

$\delta II = \frac{1}{2}\ (n-1)\ H_1\ \delta I$.

Proof

From (43) and (45) we find that the condition $\delta II = \frac{1}{2}(n-1) H_1 \delta I$ implies that

(54) $h_{ii} = hk_i^2$

sub stituting from (54) in (42) we get $H_{n-2} \Sigma k_i^2 + H_1 H_{n-2} \Sigma k_i - H_1 K \cdot (n-1) + (n-1)\{H_1 K + (n-2) H_2 H_{n-2} - (n-1) H_1^2 H_{n-2} (n+2)\} = 0$

then

$-(n^2-3n+4) H_1^2 H_{n-2} = 0$

and this leads to (53)

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