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1 DESIGN OF REDUCED ORDER OBSERVERS FOR GENERALIZED
STATE SPACE SYSTEMS CONTAINING UNKNOWN INPUTS

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4 ABSTRACT

In this paper a method is developed for the design of Luenberger-type observers for linear time-invariant control systems whose state equation is of the form $E\dot{x} = Ax + Bu + Mg$ where E is a singular matrix and g is an unknown input vector. The method is based on the singular-value decomposition of the matrix E, and on the reduction of the equation $E\dot{x}=Ax+Bu+Mg$ to a system consisting of a differential equation of form $\dot{w}_1=F_1w_1+F_2w_2+G_1u+K_1g$ and an algebraic equation of the form $H_1w_1+H_2w_2+G_2u+K_2g = 0$. If w_2 can be eliminated from the differential equation by the aid of the algebraic equation and original output equation of the system, the method yields a reduced order observer for the generalized state space system.

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INTRODUCTION

Linear control systems of the form

$$\left.\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \qquad (1)$$

as well as their discrete - time analogue

$$\left.\begin{aligned} E_{k+1}x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned}\right\} \qquad (2)$$

$k= 0,1,2,.....$

where E and E_{k+1} are singular square matrices or rectangular matrices have recently attracted considerable attention under the alternative names "singular systems","generalized state-space systems"or" descriptor systems". (C.f. the primary reference (Verghese el al.1981)).For brief discussion of discrete-time systems see Luenberger's 1977,1978).It seems that most of the research on singular systems has dealt with systems where E and A, alternatively, E_k and A_k ,are square matrices and the corresponding determinants det (sF-A),and det $(sE_k- A_k)$,where s is a scalar parameter,are not identically zero.Such system are termed"regular singular".The Problem of designing reduced order observers for ordinary time-invariant linear systems containing unknown inputs was investigated by Das and Ghoshal(1981),Miller and Mukundan (1982),Kurek(1982)and Fairman et al.(1984).However,no effort seems to have been made to develop a theory of observers for generalized state space systems with the one exception of El-Tohami's et al.(1983),which is restricted to the case whose input vector is completely known.The purpose of this paper is to develope a method for the design of reduced order observers for generalized state-space system containing unknown inputs which has the form

$$E\dot{x} = Ax + Bu + Mg \qquad (3)$$

$$y = Cx \qquad (4)$$

where E and A are q x n, B is q x p, M is qx , C is m x n, x is the

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state vector, u is the input vector, y is the output vector and g is the unmeasurable input disturbance vector.

Since this method makes use of matrix generalized inverses, a breif summary of some basic terms and notations will first be given. Let E be a mxn matrix. Any nxm matrix X satisfying the equations

$$EXE = E \tag{5}$$

$$XEX = X \tag{6}$$

$$(EX)^* = EX \tag{7}$$

$$(XE)^* = XE \tag{8}$$

is called $\{1,2,3,4\}$ inverse or a Moore-Penrose inverse(pseudoinverse) of the matrix E; denoted by M^+ or $E^{(1,2,3,4)}$. Any matrix X that satisfies (5) is called a $\{1\}$ - inverse of E and is usually denoted by $E^{(1)}$. (For the theory of all kinds of matrix generalized inverses, see Ben-Israel and Greville (1974). For an introduction to the applications of $\{1\}$- inverses in systems science, see Lovass-Nagy et al. (1978).) Early applications of matrix generalized inverses to singular (not necessarily regular) systems are discussed by Lovass-Nagy and Powers (1974, 1975). Campbell (1980) bases this treatment of time-invariant singular systems of differential equations entirely on the use of matrix generalized inverses.

The importance of the $\{1\}$- inverse lies in its application to the solution of systems of equations. Let $E^{(1)}$ be any $\{1\}$ -inverse of E. Then the equation $Ez = g$ can be solved for z if and only if $(I_m - EE^{(1)})g=0$. (I_m denotes the m x m identity matrix.) If this condition is satisfied, the general solution of $Ez = g$ is $z = E^{(1)} g + (I_n - E^{(1)} E) h$ where h is an arbitrary n x 1 vector. If E has full row rank (the rank of E equals the number of rows), then the equation $Ez = g$ can always be solved for z, for in this case $EE^{(1)} = I_m$ for all $\{1\}$-inverses of E. On the other hand, if E has full column rank (the rank of E equals the number of the columns), then the equation $Mz=g$ may or may not be solvable for z. But, if the condition $(I_m - EE^{(1)}) g = 0$ is satisfied, the solution of the equation $Ez=g$ is unique, because in this case $E^{(1)} E = I_n$ for all $\{1\}$-inverses of E.

Recently, matrix generalized inverses have been used to study a wide spectrum of problems related to singular systems, such as eigenvalue assignment (Al-Nasr et al.1983 a),output function control (Al-Nasr et al.1983 b) and solvability of discrete-time systems(Lewis 1983) . In the paper by Campbell (1983), time-varying regular singular systems are investigated by the aid of a special kind of matrix generalized inverse, the "Drazin inverse". This idea also implies certain advantages from the point of view of the actual calculations, as it is well known that " singular value decomposition ". combined with Householder transformations is an established procedure to obtain the Moore-Penrose generalized inverse of a constant matrix(Golub and Kahan 1965,Businger and Golub 1971, and Golub (1983)). In a paper by Klema and Laub(1980), various computational aspects of the "singular value decomposition " formula of the Moore-Penrose inverse have been discussed, and various applications to linear systems have been outlined.

THE METHOD

Consider the time-invariant linear systems of the form (3),(4). Let the singular values and the corresponding orthonormal sets of left and right singular vectors of the matrix E be denoted by

$$\sigma_1, \sigma_2, \ldots, \sigma_n$$

$$v_1, v_2, \ldots, v_n, \quad v_i^T v_j = \delta_{ij}$$

$$u_1, u_2, \ldots, u_q, \quad u_i^T u_j = \delta_{ij}$$

(Recall that $E^T E v_i = \sigma_i^2 v_i$, $EE^T u_i = \sigma_i^2 u_i$). If rank(E)= r, then

$\sigma_i \neq 0$ for i = 1,2, ..., r and $\sigma_i = 0$ otherwise . Let

$$V = [v_1, \ldots, v_n], \quad V^{-1} = V^T; \quad U = [u_1, \ldots, u_q], \quad U^{-1} = U^T ;$$

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$$

...

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Then, equation (3) can be written as

$$U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \dot{x} = AVV^T x + Bu + Mg \qquad (9)$$

Premultiply equation (9) by $U^{-1} = U^T$, introduce the notations $V^T x = w$ and let

$$U^T AV = S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \quad U^T = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where S_1 is r x n, S_2 is (q-r) x n, $U_1^T = \left[u_1, u_2, \ldots, u_r\right]^T$, $U_2^T = \left[u_{r+1}, u_{r+2}, \ldots, u_q\right]^T$, w_1 is r x 1, w_2 is (n-r) x 1. Equation (9) thus splits up into the following two equations

$$\dot{w}_1 = \Sigma^{-1}(S_1 w + U_1^T Bu + U_1^T Mg) \qquad (10)$$

$$S_2 w + U_2^T Bu + U_2^T Mg = 0 \qquad (11)$$

Finally, equation (4) becomes

$$y = CVw \qquad (12)$$

Partition S_1 and S_2, respectively, as

$$S_1 = \left[S_{11}, S_{12}\right] \text{ and } S_2 = \left[S_{21}, S_{22}\right]$$

where S_{11} is r x r, S_{12} = r x(n-r), S_{21} is (q-r) x r, S_{22} is (q-r) x (n-r), and partition the product matrix Q = CV as $Q = \left[Q_1, Q_2\right]$ where Q_1 is m x r and Q_2 is m x (n-r), then equations (11) and (12) can be written as

$$S_{21} w_1 + S_{22} w_2 + U_2^T Bu + U_2^T Mg = 0 \qquad (13)$$

...

. . .

and

$$Q_1 w_1 + Q_2 w_2 = y \tag{14}$$

Equations (13) and (14) can be compressed into the equation

$$Rw_2 = - Nw_1 + \gamma \tag{15}$$

where

$$R = \begin{bmatrix} S_{22} \\ \\ Q_2 \end{bmatrix}, \quad N = \begin{bmatrix} S_{21} \\ \\ Q_1 \end{bmatrix}, \quad \gamma = - \begin{bmatrix} U_2^T (Bu + Mg) \\ \\ -y \end{bmatrix}$$

Equation (15) is consistent if and only if

$$(I - RR^{(1)})(-Nw_1 + \gamma) = 0 \quad \text{for} \quad t > 0 \tag{16}$$

then, $w_2 = R^{(1)}(-Nw_1 + \gamma) + (I - R^{(1)}R)\phi$ where ϕ is an arbitrary $(n-r) \times 1$ vector. This solution yields a unique w_2 if and only if R has full column rank, i.e. $R^{(1)} R = I_{n-r}$ for any $\{1\}$- inverse of R. In what follows, it will be assumed that R has full column rank, and thus

$$w_2 = R^{(1)}(-Nw_1 + \gamma) \tag{17}$$

Substitution of equation (17) into (10) yields

$$\dot{w}_1 = Aw_1 + \hat{B} \begin{bmatrix} u \\ \\ y \end{bmatrix} + Mg \tag{18}$$

and the consistency condition (16) can be written as

$$(I-RR^{(1)})Nw_1 = (I-RR^{(1)}) \begin{bmatrix} -U_2^T B & 0 \\ & \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ \\ y \end{bmatrix} + (I-RR^{(1)}) \begin{bmatrix} -U_2^T M \\ \\ 0 \end{bmatrix} g$$

which yields

$$Kw_1 = H \begin{bmatrix} u \\ y \end{bmatrix} + \hat{H}g \tag{19}$$

where

$$K=(I-RR^{(1)})N, \quad H=(I-RR^{(1)}) \begin{bmatrix} -U_2^T B & 0 \\ 0 & I \end{bmatrix}, \quad \hat{H} =(I-RR^{(1)}) \begin{bmatrix} -U_2^T M \\ 0 \end{bmatrix}$$

Now one has to find an observer whose state space equation is

$$\dot{z} = Fz+(T\hat{B}+GH) \begin{bmatrix} u \\ y \end{bmatrix} \tag{20}$$

and $z \rightarrow Tw_1$ as $t \rightarrow \infty$ where $\hat{B}$ and H are given matrices, and the r x r matrices F and T, and the r x (q+m-r) matrix G are to be determined.

Premultiplication of (18) by T and subtraction of the resulting equation from (20) yields

$$\frac{d}{dt}(z-Tw_1) = F(z-Tw_1) + (FT-T\hat{A})\, w_1 + GH \begin{bmatrix} u \\ y \end{bmatrix} - T\hat{M}g \tag{21}$$

If F, T and G are chosen so that

$$(FT-T\hat{A})w_1 + GH \begin{bmatrix} u \\ y \end{bmatrix} - T\hat{M}g = 0 \tag{22}$$

equation (21) will reduce to

$$\frac{d}{dt}(z-Tw_1) = F(z-Tw_1) \tag{23}$$

Substitution of $H \begin{bmatrix} u \\ y \end{bmatrix} = Kw_1 - \hat{H}g$, from (19), into (22) yields

$$(GK + FT - T\hat{A})w_1 - (T\hat{M}+ G\hat{H})\, g = 0 \tag{24}$$

Since g is an unknown input, one must have

$$T\hat{M}+ G\hat{H}= 0 \tag{25}$$

Hence (24) and (25) yield

$$GK + FT - T\hat{A} = 0 \tag{26}$$

Equations (25) and (26) are equivalent to

$$\left[T\hat{M},\ T\hat{A}- FT\right] = GD \qquad (27)$$

where

$$D = \left[- \hat{H},\ K\right] .$$

Now one has to find an r x r matrix F that has prescribed eigenvalues whose real parts must be negative, a nonsingular r x r matrix T, and any r x (q+m-r)matrix G that satisfy equation (3.41).

Of course, the "nicest" result is obtained in the case where T is an identity matrix, i.e. in the case where we can find a G such that

$$\left[\hat{M},\ \hat{A}- F\right] = GD \qquad (28)$$

However, this is not always possible.

One can proceed as follows

1)Let $D^{(1)}$ be a {1 }- inverse of D. Since G will be computed from (27), one must have

$$\left[T\hat{M},\ T\hat{A}- FT\right](I - D^{(1)}D) = 0 \qquad (29)$$

Use this equation to specify some of the undetermined elements of F and T.

2) Use the still unspecified elements of T to satisfy the condition $\det(T)\neq 0$.

3) Use the still unspecified elements of F to satisfy $\det(\lambda_i I - F)= 0$ where $\lambda_i, i=1,2,\ldots,r$ are prescribed eigenvalues.

4) Compute G, where

$$G = \left[T\hat{M},\ T\hat{A}- FT\right] D^{(1)} + h(I-DD^{(1)}) \qquad (30)$$

h being arbitrary.

Example(Chua and Lin 1975,P.348)

Let

$$E=\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad A=\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, B=\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},\quad M=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

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$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The non-zero singular values of E are $\sigma_1 = \sigma_2 = \sqrt{8}$. Hence one obtains

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \ U = I_4$$

Thus

$$S = U^T A V = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, U_1^T B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, U_1^T M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ U_2^T B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_2^T M = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ and } CV = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Hence substitution in (3) yields

$$\dot{w}_1 = \frac{1}{4} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} w_1 + \frac{1}{4} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} w_2 + \frac{1}{\sqrt{8}} \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad (10')$$

and

$$\sqrt{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} g = 0 \qquad (13')$$

The output equation y = Cx becomes

$$y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} w_1 + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} w_2 \qquad (14')$$

Now, using (13') ,one can eliminate w_2 from equations (10') and (14') and obtain, respectively

$$\dot{w}_1 = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} w_1 + \frac{\sqrt{2}}{8}\left[\begin{array}{c|cc} 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]\begin{bmatrix} u \\ y \end{bmatrix} + \frac{\sqrt{2}}{8}\begin{bmatrix} -1 \\ 1 \end{bmatrix} g$$

and

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} w_1 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\begin{bmatrix} u \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} g$$

Thus, using equation (20), one can construct an observer for w_1.

In this case

$$D = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ -1 & 0 & \sqrt{2} \end{bmatrix} \text{and the matrix } \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is a } \{1\} -$$

inverse of D,one may write $I-D^{(1)}D=\text{diag}\{0,1,0\}$.

An identity observer may be constructed if there exists a 2 x 2 matrix $F=\left[f_{ij}\right]$ having prescribed eigenvalues λ_1 and λ_2 and satisfying equations(28). Hence, in equation (29)let $T = I_2$, F must be of the form

$$F=\begin{bmatrix} 0 & f_{12} \\ -\frac{1}{4} & f_{22} \end{bmatrix}$$

The elements f_{12} and f_{22} may be determined from $\det(\lambda I-F)= \lambda^2 -\lambda f_{22}+ \frac{1}{4} f_{12}$.

If $\lambda_1 \neq \lambda_2$, the equations

$$f_{12} - 4\lambda_1 f_{22} = - 4 \lambda_1^2$$

and

$$f_{12} - 4 \lambda_2 f_{22} = - 4 \lambda_2^2$$

yield f_{12} and f_{22}.

If $\lambda_1 = \lambda_2$, f_{12} and f_{22} are obtained from the equations

$$f_{12} - 4 \lambda f_{22} = - 4 \lambda^2 \quad ; \; - 4f_{22} = - 8\lambda$$

i.e. in the latter case, $f_{22} = 2\lambda$ and $f_{12} = 4\lambda^2$.

Hence, one can compute G from equation (30) where $T = I_2$.

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CONCLUSION

A method has been developed for the design of observers for linear time-invariant control systems of the form $E\dot{x} = Ax + Bu + Mg$, $y = Cx$ where E is a singular matrix and g is an unknown input vector. Such a system may or may not have a solution. If a solution exists, it can be obtained by looking for a vector that satisfies an ordinary(non-singular)matrix differential equation and an algebraic matrix equation simultaneously. The observer design method is based on the singular-value decomposition of the matrix E, which can be obtained by making use of the algorithm developed by Golub and Kahan(1965), based on Householder transformations. If the given singular differential system has any solution at all, and if a simple rank condition is satisfied, the method yields a reduced order Luenberger-type observer for the solution of the singular control problem containing unknown inputs.

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