A.M.E.

MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

" ON THE STOKESIAN MOTION OF VISCOUS LIQUIDS DUE TO AN OFF-CENTRE SOURCE "

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ABSTRACT

The Stokesian motion of viscous liquid lubricants in externally pressurized thrust collar bearings due to an off-centre pressure source has been studied by potential theory. The complex potential function that satisfies the boundary conditions of the bearing has been constructed by reflection across the ring circles. It has been determined in terms of Jacobi Theta Functions. The pressure distribution has been obtained for the following cases:-

i- Externally pressurized thrust circular concentric collar bearings.

ii- Externally pressurized thrust elliptic collar bearings.

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1. INTRODUCTION.

The study of the Stokesian motion of viscous fluids finds an important application in the design and analysis of externally pressurized hydrostatic bearings [1 - 5]. A class of these bearings that is characterized by having large recesses, gives good load carrying capacity. On the other hand, it has been reported (Cf.[6]) that this class possesses poor stability when such bearings are subjected to cyclic vibrations. To remedy this deficiency , it was thought better to use thrust collar bearings having fluid supply holes that communicate directly with the lubricant film , thereby avoiding recesses.

In the present paper , the Stokesian motion of viscous liquid lubricants in thrust collar bearings is studied theoretically. These bearings are without recesses and they are externally pressurized by an off-centre supply source. Two geometries, namely, the circular and elliptic bearings have been investigated.

2. MATHEMATICAL MODEL.

Figure (1) shows a general configuration of a thrust collar bearing. The boundary ∂D_1 denotes the thrust collar while ∂D_2 is the bearing outer edge. The pressure source is situated at the point A . For simplicity, A is taken on the x - axis.

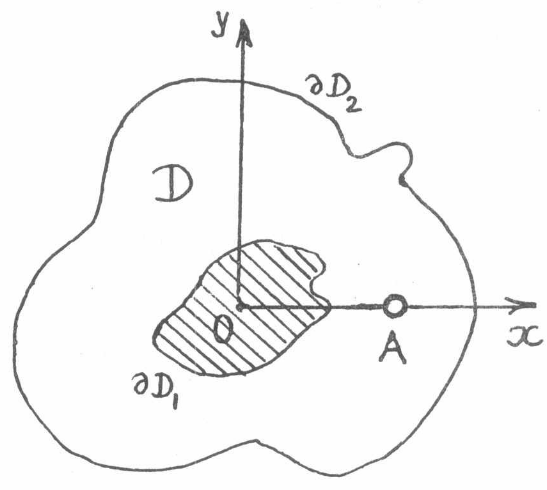

Fig.(1) Bearing Configuration.

It is to be noted that the boundaries ∂D_1 and ∂D_2 are composed of piecewise smooth arcs. This condition is always realized in practical bearings configurations. To study the Stokesian motion of the lubricant in the domain D that is bounded by ∂D_1 and ∂D_2 boundaries , one introduces the vectorial quantity $\vec{Q}$ that indicates the volume flow rate

. . .

per unit length. Following Ref.[1] , we write

$$Q_x = - \frac{h^3}{12\mu} \frac{\partial p}{\partial x} \tag{2.1a}$$

$$Q_y = - \frac{h^3}{12\mu} \frac{\partial p}{\partial y} \tag{2.1b}$$

in which p is the liquid pressure , h is the film thickness and μ is the coefficient of viscosity.

From the continuity equation,i.e,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{2.2}$$

we introduce the Likewise Stream Function $\Omega(x,y)$ which is defined by

$$Q_x = - \frac{h^3}{12\mu} \frac{\partial \Omega}{\partial y} \tag{2.3a}$$

$$Q_y = \frac{h^3}{12\mu} \frac{\partial \Omega}{\partial x} \tag{2.3b}$$

It is clear that Eqs. (2.1 & 2.3) reveal that p and Ω are harmonic conjugate functions.
This last statement leads us to put forward the function $w = p + i\Omega$ where w is denoted as the complex potential. It is an analytic function of the variable $z = x + i\,y$.

Thus,the problem under consideration is reduced to the determination of the analytic function $w(z)$ that satisfies the following conditions :

$$\mathbb{R}(w) = 0 \qquad \text{for} \quad z \in (\partial D_1 \cup \partial D_2) \tag{2.4a}$$

$$\mathbb{R}(w) = p_s > 0 \qquad \text{for} \quad |z - \ell| = r_s \tag{2.4b}$$

in which ℓ denotes the distance OA , r_s is the radius of the source and p_s is the pressure at the source.

3. COMPLEX POTENTIAL.

To determine the complex potential $w(z)$ that satisfies Eqs.(2.4a&b), we first consider the annulus B as a canonical domain. The inner circle bounding B has a radius r_1 while the radius of the outer circle bounding B is r_2 as shown in Fig. (2) .

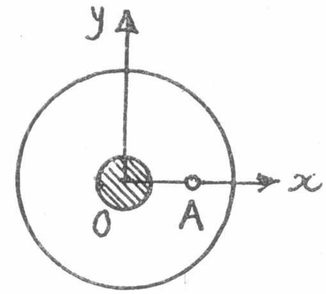

Fig.(2) Canonical Bearing.

Now, we define the Hyper-potential Function $W(z)$ as :

$$W(z) = \exp(-w) \tag{3.1}$$

that satisfies

$$|W| = 1 \qquad \text{for } |z| = r_1 \text{ and } |z| = r_2 \tag{3.2}$$

and

$$\ln |W| = -p_s \qquad \text{for } |z-l| = r_s \tag{3.3}$$

We attempt to continue the Hyper-potential $W(z)$ beyond the two circles. To achieve this end , we associate to each point $z \in B$, a corresponding point z_1 inside the inner circle such that $z z_1 = r_1^2$. Thus, $W(z)$ satisfies the functional equation

$$W(z) \cdot W(r_1^2/z) = 1 \tag{3.4}$$

Similarly, reflection across the outer circle leads to the second functional equation

$$W(z) \cdot W(r_2^2/z) = 1 \tag{3.5}$$

Since the point A is a simple zero of $W(z)$, it follows from successive application of Eqs.(3.4 & 5) that $W(z)$ has simple zeros at the points :

$$l \; ; \; r_1^2 r_2^{-2} l \; ; \; r_1^{-2} r_2^2 l \; ; \; \ldots . \; ; \; r_1^{2n} \; r_2^{-2n} l \; ; \; r_1^{-2n} \; r_2^{2n} l, \; \ldots .$$

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and simple poles at the points :

$r_1^2 \ell^{-1}$; $r_2^2 \ell^{-1}$; $r_1^4 r_2^{-2} \ell^{-1}$; $r_1^{-2} \quad r_2^4 \ \ell^{-1}$; .... ; $r_1^{2n} \ r_2^{-2n+2} \ell^{-1}$

$r_1^{-2n+2} \quad r_2^{2n} \ell^{-1}$; ....

Thus, a function with the zeros and poles aforementioned may take the form

$$F(z) = \left(1 - \frac{z}{\ell}\right) \frac{\prod_{n=1}^{\infty} \left(1 - r_1^{2n} r_2^{-2n} \frac{z}{\ell}\right)\left(1 - r_1^{2n} r_2^{-2n} \frac{\ell}{z}\right)}{\prod_{n=1}^{\infty} \left(1 - r_1^{2n-2} r_2^{-2n} \ell z\right)\left(1 - r_1^{2n} r_2^{-2n+2} \frac{1}{\ell z}\right)} \tag{3.6}$$

A direct calculation reveals that the function $F(z)$ satisfies the following equations :

$$F(z) \cdot F(r_1^2/z) = 1 \tag{3.7}$$

$$F(z) \cdot F(r_2^2/z) = (r_2/\ell)^2 \tag{3.8}$$

Taking tentatively $W(z) = \alpha z^{\beta} F(z)$ where α and β are constants to be determined from substitution in Eqs. (3.4 & 5) , we get

$$\alpha = -\ell^{\frac{1}{2}} \left[\frac{1}{(r_1^2 r_2^2)^{\beta} \ r_2^2} \right]^{\frac{1}{4}} \tag{3.9}$$

and

$$\beta = \frac{\ln (r_2/\ell)}{\ln (r_1/r_2)} \tag{3.10}$$

Defining the "nome" $\lambda = r_1 / r_2$ and taking logarithms for both sides of Eq. (3.1), the complex potential w is given by

$$
\begin{aligned}
-w(z) = {} & \ln\alpha + \beta \ln z + \ln(1 - z/\ell) + \\
& + \sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} z/\ell\right) - \sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} r_2^2/\ell z\right) + \\
& + \sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} \ell/z\right) - \sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} \ell z/r_1^2\right)
\end{aligned}
\tag{3.11}
$$

Now, we consider the Jacobian Theta Functions (Cf.[7] ,p.354) $\Theta_1(Z,\lambda)$ and $\Theta_4(Z,\lambda)$:

$$
\begin{aligned}
\Theta_1(Z,\lambda) &= -i\,G\lambda^{\frac{1}{4}}\left(e^{iZ} - e^{-iZ}\right) \\
& \quad \prod_{n=1}^{\infty}\left(1 - \lambda^{2n} e^{2iZ}\right)\left(1 - \lambda^{2n} e^{-2iZ}\right) \\
&= 2G\lambda^{\frac{1}{4}} \sin Z \,.\prod_{n=1}^{\infty}\left(1 - \lambda^{2n}\cos 2Z + \lambda^{4n}\right)
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\Theta_4(Z,\lambda) &= G\prod_{n=1}^{\infty}\left(1 - \lambda^{2n-1} e^{2iZ}\right)\left(1 - \lambda^{2n-1} e^{-2iZ}\right) \\
&= G\prod_{n=1}^{\infty}\left(1 - 2\lambda^{2n-1}\cos 2Z + \lambda^{4n-2}\right)
\end{aligned}
\tag{3.13}
$$

in which

$$
G = \prod_{n=1}^{\infty}\left(1 - \lambda^{2n}\right) \tag{3.14}
$$

Letting $z/\ell = e^{2iZ}$, substituting into Eq.(3.11) and noting that

$$
\sum_{n=0}^{\infty} \ln\left(1 - \lambda^{2n} z/\ell\right) + \sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} \ell/z\right) =
$$

$$
\ln\left[\frac{\Theta_1\left(\frac{1}{2i}\ln z/\ell, \lambda\right)}{i\,G\,\lambda^{\frac{1}{4}}\,(\ell/z)^{\frac{1}{2}}}\right] \tag{3.15}
$$

and

$$
\sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} r_2^2/\ell z\right) + \sum_{n=1}^{\infty} \ln\left(1 - \lambda^{2n} \ell z/r_1^2\right) =
$$

$$
\ln\left[\frac{\Theta_4\left(\frac{1}{2i}\ln(\ell z/r_1 r_2), \lambda\right)}{G}\right] \tag{3.16}
$$

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we obtain

$$w(z) = \ln \left[i \frac{\lambda^{\frac{1}{4}} \ell^{\frac{1}{2}}}{\alpha z^{\beta+\frac{1}{2}}} \frac{\Theta_4 \left(\frac{1}{2i} \ln \ell z / r_1 r_2 , \lambda \right)}{\Theta_1 \left(\frac{1}{2i} \ln z / \ell , \lambda \right)} \right] \qquad (3.17)$$

4. PRESSURE DISTRIBUTION.

Case I. Circular Bearing :

For the determination of the pressure distribution inside a circular thrust collar bearing subject to the conditions Eqs.(2.4 a & b) , the real part of Eq.(3.17) should be evaluated. This can be easily achieved by noting the relations between Theta and Circular Functions via Schottky Functions (Cf. [8]). Thus, Eq.(3.17) is simplified to take the form :

$$w(z) = \ln \frac{\sin\left(\pi/2\rho \, \ln \ell z / r_1^2 \right)}{\sin\left(\pi/2\rho \, \ln z/\ell \right)} \qquad (4.1)$$

in which $\rho = \ln \lambda$

Therefore, the pressure p which is $\mathbb{R}$ (w) can be expressed as

$$p = K \ln \left(\frac{\sin^2 A + \operatorname{Sinh}^2 B}{\sin^2 C + \operatorname{Sinh}^2 B} \right) \qquad (4.2)$$

in which

$$A = \frac{\pi}{2\rho} \ln \frac{\ell r}{r_1^2} \qquad (4.3)$$

$$r = (x^2+y^2)^{\frac{1}{2}} = |z| \qquad (4.4)$$

$$B = \frac{\pi}{2\rho} \arg.(z) \qquad (4.5)$$

$$C = \frac{\pi}{2\rho} \ln \frac{r}{\ell} \qquad (4.6)$$

We note that the scale constant K is introduced in Eq.(4.2) simply to satisfy the condition Eq.(2.4b).

Substituting $p = p_s$ for $r = \ell - r_s$ and $\arg(z) = 0$, the value of the constant K is given by

$$K = \frac{p_s}{\ln(\sin^2 A_s) - \ln(\sin^2 C_s)} \qquad (4.7)$$

in which

$$A_s = \frac{\pi}{2\rho} \ln \ell(\ell - r_s)/r_1^2 \qquad (4.8)$$

$$C_s = \frac{\pi}{2\rho} \ln (\ell - r_s)/\ell \qquad (4.9)$$

Case II. Elliptic Bearing :

We consider the elliptic thrust collar bearing shown in Fig.(3). The collar boundary ∂D_1 is given by :

$$f^2 (z^2 + z^{*2}) - 2(a^2+b^2) z z^* + 4 a^2 b^2 = 0 \qquad (4.10)$$

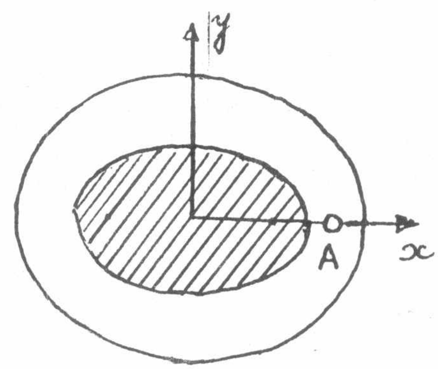

Fig.(3) Elliptic Bearing.

in which $2a$ and $2b$ are the major and minor-axes of the collar respectively.
f is the focal distance and it is equal to $\sqrt{a^2 - b^2}$.
The asterisk denotes the conjugate of a complex quantity.

The bearing outer edge ∂D_2 takes the form :

$$f^2 (z^2 + z^{*2}) - 2(c^2+d^2) z z^* + 4c^2 d^2 = 0 \qquad (4.11)$$

in which $2c$ and $2d$ are the major and minor-axes of the outer edge. The pressure source is situated at A as shown in Fig.(3) at a distance ℓ from O in which ℓ satisfies the inequalities $a < \ell < c$.

To determine the pressure distribution in the domain D of the bearing via the theory of Sec.3 , we must map the domain D in the z - plane onto the annulus B in the ζ - plane. This mapping can be accomplished in two steps. Firstly, we map D conformally onto an infinite strip in the intermediate t - plane by the relation

$$t = \mathrm{Cosh}^{-1}\left(z/\sqrt{c^2-d^2}\right) \tag{4.12}$$

Setting $t = u + i\,v$, then from Eq.(4.12) we obtain

$$x = \sqrt{c^2-d^2}\ \mathrm{Cosh}\,u\ \cos v \tag{4.13a}$$

$$y = \sqrt{c^2-d^2}\ \mathrm{Sinh}\,u\ \sin v \tag{4.13b}$$

Further, the transformation

$$\zeta = \sqrt{\frac{c-d}{c+d}}\ \exp\,(t) \tag{4.14}$$

maps the strip in the t - plane onto the annulus B in the ζ - plane such that its outer radius is unity and its inner radius λ is equal to $\frac{a+b}{c+d}$. Now, the pressure source lies on the horizontal axis ξ in the ζ - plane at a distance from O given by

$$\ell_1 = \frac{\left(\ell + \sqrt{\ell^2 - c^2 + d^2}\right)}{c+d} \tag{4.15}$$

Substituting for ζ from Eq.(4.14) into Eq.(4.1) - note that z is now replaced by ζ in Eq.(4.1) -, we get

$$w = \ln \frac{\sin \frac{\pi}{2\rho}(t+e)}{\sin \frac{\pi}{2\rho}(t+h)} \tag{4.16}$$

in which the constants e and h are

$$e = \frac{1}{2}\ln\frac{c-d}{c+d} + \ln \ell_1 - 2\ln\frac{a+b}{c+d} \tag{4.17}$$

$$h = \frac{1}{2}\ln\frac{c-d}{c+d} - \ln \ell_1 \tag{4.18}$$

Thus, the pressure p which is $\mathbb{R}(w)$ can be written

$$p = K \ln \left(\frac{\sin^2 D + \operatorname{Sinh}^2 E}{\sin^2 F + \operatorname{Sinh}^2 E} \right) \tag{4.19}$$

in which

$$D = \frac{\pi}{2\rho}(u+e) \tag{4.20}$$

$$E = \frac{\pi}{2\rho} v \tag{4.21}$$

$$F = \frac{\pi}{2\rho}(u+h) \tag{4.22}$$

The scale constant K is evaluated by substituting $u_1 = \operatorname{Cosh}^{-1} \frac{l - r_s}{\sqrt{c^2 - d^2}}$, $v_1 = 0$ for $p = p_s$ in Eq.(4.19) .
This gives

$$K = \frac{p_s}{\ln\left[\sin^2 \frac{\pi}{2\rho}(u_1+e)\right] - \ln\left[\sin^2 \frac{\pi}{2\rho}(u_1+h)\right]} \tag{4.23}$$

5. CONCLUSIONS.

Methods of potential theory have been applied for the analysis of externally pressurized thrust collar bearings, The pressure distribution has been determined for circular and elliptic bearings for a pressure source that communicates directly with the lubricant film without feeding into recesses. Also, the results obtained can be directly utilized in the design of bearings with lubricants that are compressible fluids. Naturally, this can be achieved by simply supplementing the polytropic exponent "γ" of the compressible lubricant and then substituting for the pressure by $P = p^{\gamma/1+\gamma}$.

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NOMENCLATURE

a , b	=	semi-major and minor axes of the elliptic collar
c , d	=	semi-major and minor axes of the outer elliptic bearing edge
f	=	focal distance
h	=	film thickness
ℓ	=	distance of the source from the bearing centre
p	=	pressure
$\vec{Q}$	=	volume flow rate per unit length vector
r_1, r_2	=	radii of the collar and the outer circular bearing edge
r_s	=	radius of the source
w	=	complex potential function
W	=	hyper-potential function
γ	=	polytropic exponent
λ	=	mapping modulus
μ	=	absolute coefficient of viscosity
Ω	=	likewise stream function