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COMPLETE FORCE-BALANCING AND SHAKING MOMENT OPTIMIZATION OF THE RGGR SPATIAL MECHANISM

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ABSTRACT

An RGGR spatial mechanism with constant input speed and arbitrary link dimensions is considered. The coupler is a thin straight rod. The masses of the moving links are distributed so that the centre of their total mass is stationary throughout the mechanism cycle to insure that the resultant shaking force transmitted to ground due to the inertia effects on the moving links vanishes. This is achieved by fixing two balancing weights to the input and output links. The balancing weight attached to the ouput link is a solid cylinder with the greatest feasible length, and with the axis of rotation of the output link coinciding with one of the generators of its cylinderical surface. The parallel plane sides of this balancing weight are, in general, inclined to its axis. Both balancing weights are so designed that the Root-Mean-Square shaking moment arising from the force-balanced mechanism is minimum. Anumerical example is presented.

INTRODUCTION

Force-balancing of a mechanism means that the forces transmitted to ground due to the inertia effects of the moving links are balanced or reduced to a pure couple. This is accomplished by distributing the masses of the moving links in such a manner that the centre of their total mass remains stationary during the mechanism motion. Berkof and Lowen [1] developed the method of linearly independent vectors for complete balance of the shaking force resulting from the inertia effects in planar mechanism including revolute pairs only. Tepper and Lowen [2] derived the conditions necessary for the application of this method to planar mechanisms containing both revolute and prismatic pairs. When a mechanism is force-balanced, the forces transmitted to ground, in general, are reduced to a pure couple called shaking moment. The Root-Mean-Square shaking moment of a force-balanced four-bar planar mechanism was minimized by Berkof and Lowen [3,4], Carson [5] and Hains [6]. So for as the author is aware, force balancing with shaking moment minimization has not been applied to spatial mechanisms.

In the present paper, the principle used in the development of the method of linearly independent vectors is applied for complete balance of the shaking

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force of an RGGR mechanism with arbitrary link dimensions. It is shown that, if the coupler is a straight thin rod, the mechanism can be force-balanced by attaching two balancing weights to the input and output links. The balancing weights are so designed that the RMS shaking moment of the force balanced mechanism is minimum.

MECHANISM DESCRIPTION AND COORDINATE SYSTEMS

Fig. 1 represents an RGGR mechanism with arbitrary link dimensions. The input crank AB, which is rotating at a uniform speed ω , and the output link DC are connected to ground through the Revolute pairs at A and D. The coupler BC is connected to the input and output links by the Globular (Spherical) pairs at B and C. The axes of rotation of the input and output links are along the lines AE and DO respectively, and the line OE is their common normal. The lines BQ_1 and CQ_3 are perpendicular to lines EA and OD respectively. Points G_1 and G_3 are the mass centres of the input crank and output link after attaching a balancing weight to each of them. Point G_2 is the coupler mass centre.

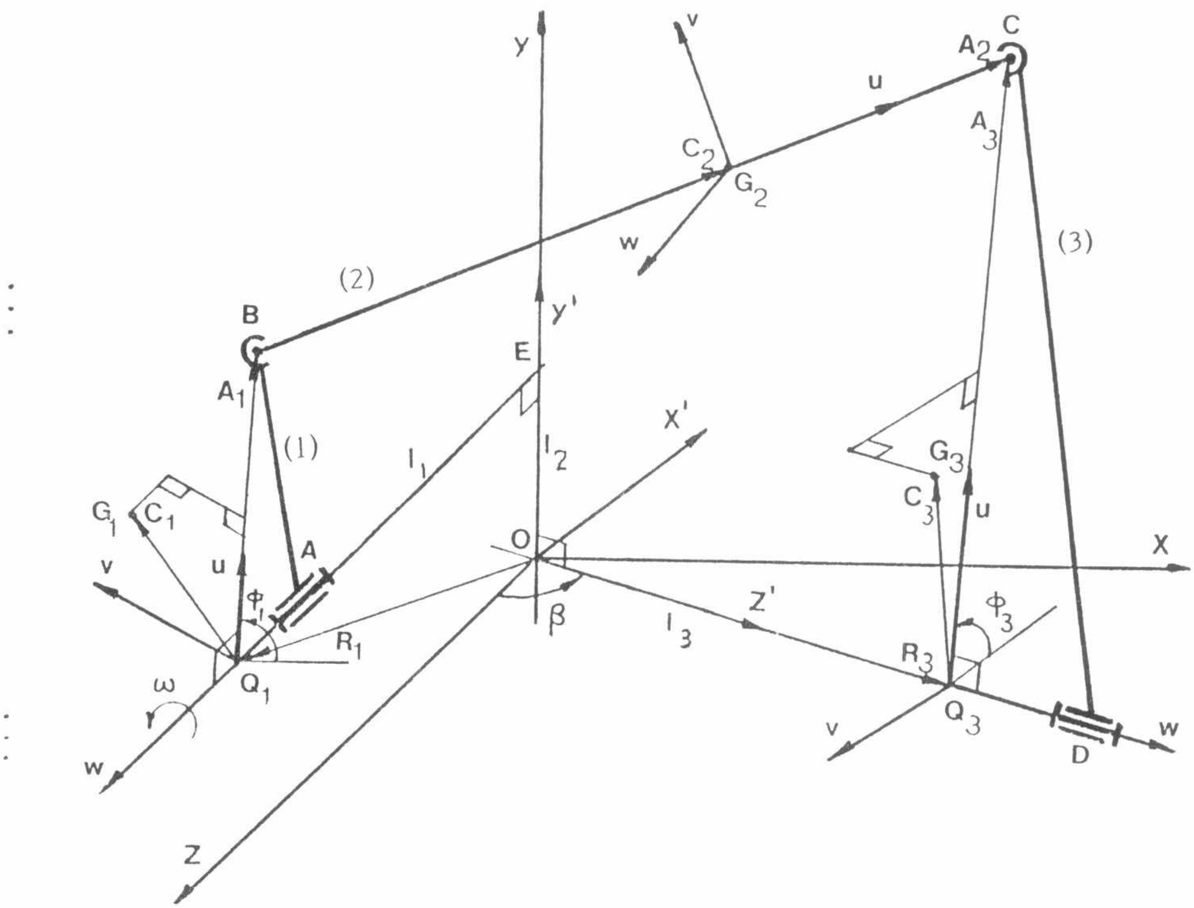

Fig. 1: An RGGR mechanism with arbitrary link dimensions.

The main space-fixed coordinate system Oxyz is located such that point E lies on the y-axis, and point Q_1 lies in the yz-plane and has a nonnegative coordinate in the z-direction. In addition to this coordinate system there is the auxiliary

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space-fixed coordinate system Ox'y'z' in which Oy' coincides with O_y and $O_{z'}$ is along the direction OQ_3. The mechanism motion is characterized by the parameters a_1,a_2,a_3,l_1,l_3,l_2 and β, which represent the lengthes of the lines Q_1B, BC, Q_3C, EQ_1 and OQ_3, the y-coordinate of point E, and the angle between the line OD and the z-axis respectively. The angular position of the input crank is defined by Ψ_1, which is the angle between the direction Q_1B and the x-axis, whereas the angular position of the ouput link is specified by Ψ_3, which is the angle between the direction Q_3C and the x'-axis.

Application of Kutzbach's formula shows that the RGGR mechanism has two degrees of freedom. In fact one of these degrees represents the idle freedom of the coupler, namely its freedom to spin about the line BC without affecting the input-output relationship. In the mechanism under consideration the coupler is assumed to be straight rod with small cross-section, and therefore the unconstrained spin of the coupler has negligible effect on the mechanism dynamics. In order to take the advantage of this characteristic, the spin of the coupler about the line BC, in the present work, is assumed to be constrained by a certain condition which facilitates the kinematic analysis of the coupler motion.

The body-fixed coordinate systems are located as follows:
1. In the coordinate system Q_1uvw, which are fixed to the input crank, the w-axis coincides with its axis of rotation and the u-axis is along the line Q_1B.
2. The u-axis of the coordinate system G_2uvw, which is fixed to the coupler, is along the line G_2C. The v-axis is assumed to be always parallel to the xy-plane. This assumption constrains the spin of the coupler about the line BC, but without affecting the mechanism dynamics.
3. In the coordinate system Q_3uvw, which is fixed to the output link, the w-axis coincides with its axis of rotation and the u-axis is along the line Q_3C.

FORCE-BALANCING

The position of the centre of the total mass of the moving links with respect to the origin O is defined by the position vector R_g (not shown in Fig. 1), which is found from:

$$(m_1+m_2+m_3)R_g=m_1(R_1+C_1)+m_2(R_1+A_1+\frac{c_2}{a_2}A_2)+m_3(R_3+C_3) \qquad (1)$$

where: m_i = mass of link (i) after the addition of the balancing weights,

R_1,R_3 = the vectors defining the positions of points Q_1 and Q_3 relative to the origin O,

A_1,A_2 = the vectors specifying the positions of points B and C with respect to points Q_1 and B respectively,

C_1,C_3 = the vectors describing the positions of the mass centres G_1 and G_3 with respect to points Q_1 and Q_3 respectively,

c_2 = length of BG_2.

The mechanism loop equation may be written as:

$$R_1 + A_1 + A_2 - A_3 - R_3 = 0, \qquad (2)$$

where A_3 is the vector defining the position of point C relative to point Q_3. Elimination of A_2 between eqs. (1) and (2) yields:

$$R_g = \frac{1}{(m_1+m_2+m_3)} \Big\{ [m_1 + \frac{m_2(a_2-c_2)}{a_2}] R_1 + [m_3 + \frac{m_2 c_2}{a_2}] R_3 + [m_1 C_1 + \frac{m_2(a_2-c_2)}{a_2} A_1] + [m_3 C_3 + \frac{m_2 c_2}{a_2} A_3] \Big\} \quad (3)$$

This equation describes the trajectory of the centre of the total mass of the moving links in terms of the stationary vectors R_1 and R_3, together with the moving vectors A_1, C_1, A_3 and C_3. The shaking force is completely balanced if the link masses are distributed in such a manner that the centre of the total mass of the moving links remains stationary throughout a complete mechanism cycle, i.e. if R_g is constant. This requires that the coefficients of the moving vectors in eq.(3) are so adjusted that the quantities inside the last two pairs of square brackets in this equation vanish. This can be achieved only if each of these quantities vanishes individually, since the vectors A_1 and C_1, which rotate with the input crank, are linearly independent of the vectors A_3 and C_3, which move with the output link. Thus, the conditions of complete balance of the shaking force may be expressed as:

$$m_1 C_1 + \frac{m_2(a_2-c_2)}{a_2} A_1 = 0 \ , \quad (4)$$

$$m_3 C_3 + \frac{m_2 c_2}{a_2} A_3 = 0 \quad (5)$$

These conditions may be fulfiled if the masses of the balancing weights, which are attached to links (1) and (3), and the body-fixed coordinates of their mass centres are related by:

$$m_{b1}\, u_{b1} = -m_{o1}\, u_{o1} - m_2\, a_1\, (1-\mu) \ , \quad (6)$$

$$m_{b1}\, v_{b1} = -m_{o1}\, v_{o1} \ , \quad (7)$$

$$m_{b1}\, w_{b1} = -m_{o1}\, w_{o1} \ , \quad (8)$$

$$m_{b3}\, u_{b3} = -m_{o3}\, u_{o3} - m_2\, a_3\, \mu \ , \quad (9)$$

$$m_{b3}\, v_{b3} = -m_{o3}\, v_{o3} \ , \quad (10)$$

$$m_{b3}\, w_{b3} = -m_{o3}\, w_{o3} \ . \quad (11)$$

where: $$\mu = \frac{c_a}{a_2} \ , \quad (12)$$

m_{o1}, m_{o3} = masses of links (1) and (3) without their balancing weights,

m_{b1}, m_{b3} = masses of the balancing weights attached to links (1) and (3),

$u_{o1}, v_{o1}, w_{o1}, u_{o3}, v_{o3}, w_{o3}$ = body-fixed coordinates of the mass centres of links (1) and (3) before the addition of the balancing weights,

$u_{b1}, v_{b1}, w_{b1}, u_{b3}, v_{b3}, w_{b3}$ = body-fixed coordinates of the mass centres of the balancing weights fixed to links (1) and (3).

It is to be noticied that, the addition of the balancing weights increases the moments of inertia of links (1) and (3) about their axes of rotation. In the mechamism under consideration the moment of inertia of link (1) about its axis of rotation has no influence on the resulting shaking moment, since it is rotating at a constant speed. In order to minimize the increase in the component of the inertia couple of link (3) along its axis of rotation, which results from the increase of its to it is taken in the form of a solid circular cylinder with the greatest feasible length, and with the w-axis coinciding with one of the generators of its cylindrical surface. Fig. 2 shows the balancing weight fixed to link (3), which is a cylinder of radius r and length L with its mass centre at point B_3. The parallel plane sides of this balancing weight make an angle α with its axis. These surfaces, in general, have elliptical form. The plane defined by the major axes of these ellipses together with the axis of the balancing weight makes an angle γ with the uw-plane fixed to link (3). The mass of this balancing weight is related to the products $m_{b3}\, u_{b3}$ and $m_{b3}\, v_{b3}$ by:

$$m_{b3} = \left\{\pi \rho L[(m_3\, u_{b3})^2 + (m_{b3}\, v_{b3})^2]\right\}^{1/3} \qquad (13)$$

where ρ is the density of the balancing weight.

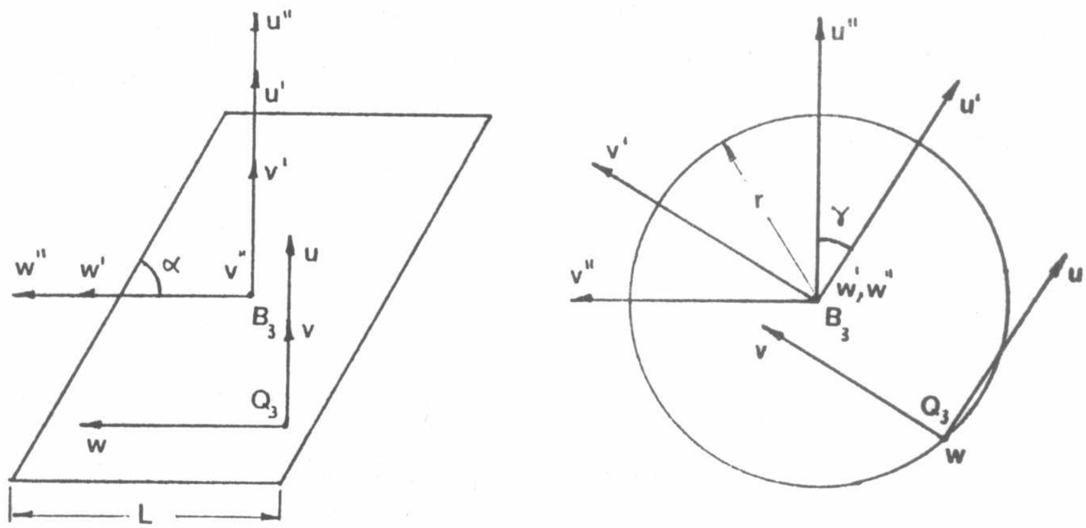

Fig. 2: Balancing weight fixed to the output link.

The position of the balancing weight attached to link (1) relative to its link may be determined from eqs. (6) to (8) by a reasonable assumption for its mass. In order to specify the mass of the balancing weight fixed to link (3) and its position relative to the link, the products $m_{b3}\, u_{b3}$ and $m_{b3}\, v_{b3}$, which are obtained from eqs. (9) and (10), are substituted into eq. (13) to determine the mass of the balancing weight, and then the position of its mass centre relative to the link is determined from eqs. (9) to (11), by the use of the calculated value of m_{b3}. The radius of the balancing weight is obtained from:

$$r = \sqrt{u_{b3}^2 + v_{b3}^2}\,. \qquad (14)$$

More information about the balancing weight fixed to link (1), as well as the angles α and γ, which are required for specifying the balancing weight attached to link (3) completely, are obtained from the conditions necessary for minimizing the RMS shaking moment.

KINEMATIC ANALYSIS

POSITION

The components of R_1 and A_1 along the axes of the Oxyz coordinate system are obtained from:

$$[x_{q1}, y_{q1}, z_{q1}]^T = [0, l_2, l_1]^T , \tag{15}$$

$$[A_{1x}, A_{1y}, A_{1z}]^T = [a_1 \cos\Phi_1, a_1 \sin\Phi_1, 0]^T. \tag{16}$$

Similarly, the components of R_3 and A_3 along the axes of the Ox'y'z' coordinate system are obtained from:

$$[x'_{q3}, y'_{q3}, z'_{q3}]^T = [0, 0, l_3]^T , \tag{17}$$

$$[A_{3x'}, A_{3y'}, A_{3z'}]^T = [a_3 \cos\Phi_3, a_3 \sin\Phi_3, 0]^T. \tag{18}$$

The components in eq. (18) are related to the components of A_3 along the main directions by:

$$\begin{bmatrix} A_{3x} \\ A_{3y} \\ A_{3z} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} A_{3x'} \\ A_{3y'} \\ A_{3z'} \end{bmatrix}. \tag{19}$$

Eqs. (18) and (19) yield:

$$[A_{3x}, A_{3y}, A_{3z}]^T = [a_3 \cos\beta \cos\Phi_3, a_3 \sin\Phi_3, -a_3 \sin\beta \cos\Phi_3]^T. \tag{20}$$

When the vectors in eq. (2) are resolved along the main directions, A_2 may be expressed, by the use of eqs. (15) to (17) and (20), as:

$$\begin{bmatrix} A_{2x} \\ A_{2y} \\ A_{2z} \end{bmatrix} = \begin{bmatrix} -a_1 \cos\Phi_1 + a_3 \cos\beta \cos\Phi_3 + l_3 \sin\beta \\ -a_1 \sin\Phi_1 + a_3 \sin\Phi_3 - l_2 \\ -a_3 \sin\beta \cos\Phi_3 + l_3 \cos\beta - l_1 \end{bmatrix}. \tag{21}$$

The components of A_2 along the main directions are related to each other by:

$$A_{2z}^2 + A_{2y}^2 + A_{2z}^2 = a_2^2 \tag{22}$$

By substituting the components of A_2 in the foregoing equation by the expressions given in eq. (21), the relation between the angular positions of the input and output links may be expressed in the form:

$$D_1 + D_2 \cos\Phi_3 + D_3 \sin\Phi_3 = 0 , \tag{23}$$

where:

$$D_1 = l_1^2 + l_2^2 + l_3^2 - 2l_1 l_3 \cos\beta + a_1^2 - a_2^2 + a_3^2 - 2l_3 a_1 \sin\beta \cos\Phi_1 + 2l_2 a_1 \sin\Phi_1 , \tag{24}$$

$$D_2 = 2a_3 (l_1 \sin\beta - a_1 \cos\beta \cos\Phi_1) , \tag{25}$$

$$D_3 = -2a_3 (l_2 + a_1 \sin\Phi_1). \tag{26}$$

When the angular position of the input crank is known, the variables D_1, D_2 and D_3 can be determined from eqs. (24) to (26), and accordingly eq. (23) may be solved for Φ_3. The solution, in general, yields two values for Φ_3. The suitable value is chosen according to the initial mechanism position.

The position of the mass centre G_1 relative to the origin O is defined by R_1+C_1, and therefore its coordinates in the Oxyz coordinate system are obtained from:

$$\begin{bmatrix} x_{g1} \\ y_{g1} \\ z_{g1} \end{bmatrix} = \begin{bmatrix} 0 \\ l_2 \\ l_1 \end{bmatrix} + \begin{bmatrix} \cos\psi_1 & -\sin\psi_1 & 0 \\ \sin\psi_1 & \cos\psi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{g1} \\ v_{g1} \\ w_{g1} \end{bmatrix} = \begin{bmatrix} u_{g1}\cos\psi_1 - v_{g1}\sin\psi_1 \\ l_2 + u_{g1}\sin\psi_1 + v_{g1}\cos\psi_1 \\ l_1 + w_{g1} \end{bmatrix}. \tag{27}$$

Similarly, the position of the mass centre G_3 with respect to the origin O is specified by $R_3 + C_3$, and its coordinates in the Ox'y'z' coordinate system are obtained from:

$$\begin{bmatrix} x'_{g3} \\ y'_{g3} \\ z'_{g3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_3 \end{bmatrix} + \begin{bmatrix} \cos\psi_3 & -\sin\psi_3 & 0 \\ \sin\psi_3 & \cos\psi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{g3} \\ v_{g3} \\ w_{g3} \end{bmatrix} = \begin{bmatrix} u_{g3}\cos\psi_3 - v_{g3}\sin\psi_3 \\ u_{g3}\sin\psi_3 + v_{g3}\cos\psi_3 \\ l_3 + w_{g3} \end{bmatrix}. \tag{28}$$

The position of the mass centre G_2 relative to the origin O is described by $R_1+A_1+\mu A_2$, and therefore its coordinates in the main coordinate system are obtained from:

$$\begin{bmatrix} x_{g2} \\ y_{g2} \\ z_{g2} \end{bmatrix} = \begin{bmatrix} (1-\mu)\,a_1\cos\psi_1 + \mu(l_3\sin\beta + a_3\cos\beta\cos\psi_3) \\ (1-\mu)(l_2 + a_1\sin\psi_1) + \mu\, a_3\sin\psi_3 \\ (1-\mu)\, l_1 + \mu(l_3\cos\beta - a_3\sin\beta\cos\psi_3) \end{bmatrix} \tag{29}$$

VELOCITY

By differentiating eq. (23) with respect to time, the angular velocity of the output link may be written in the form:

$$\dot{\psi}_3 = \frac{D_4 + D_5\cos\psi_3 + D_6\sin\psi_3}{D_2\sin\psi_3 - D_3\cos\psi_3}, \tag{30}$$

where:

$$D_4 = 2a_1\omega\,(l_2\cos\psi_1 + l_3\sin\beta\sin\psi_1), \tag{31}$$

$$D_5 = 2a_1\, a_3\omega\cos\beta\sin\psi_1, \tag{32}$$

$$D_6 = -2a_1\, a_3\omega\cos\psi_1\,. \tag{33}$$

The velocity of point C relative to point B is given by $\dot{A}_2$, and is expressed, by differentiating eq. (21) with respect to time, as:

$$\begin{bmatrix} \dot{A}_{2x} \\ \dot{A}_{2y} \\ \dot{A}_{2z} \end{bmatrix} = \begin{bmatrix} a_1\omega\sin\psi_1 - a_3\dot{\psi}_3\cos\beta\sin\psi_3 \\ -a_1\omega\cos\psi_1 + a_3\dot{\psi}_3\cos\psi_3 \\ a_3\dot{\psi}_3\sin\beta\sin\psi_3 \end{bmatrix}. \tag{34}$$

This velocity can also be obtained from:

$$\dot{A}_2 = \Omega \times A_2\,, \tag{35}$$

where Ω is the angular velocity of the coupler. Resolution of the vectors in this equation along the axes of the coordinate system G_2uvw yields:

$$[(\dot{A}_2)_u, (\dot{A}_2)_v, (\dot{A}_2)_w]^T = [0,\ a_2\Omega_w,\ a_2\Omega_v]^T\,. \tag{36}$$

The relation between the components of A_2 along the axes of the Oxyz and G_2uvw coordinate systems may be written in the form:

$$\begin{bmatrix} \dot{A}_{2x} \\ \dot{A}_{2y} \\ \dot{A}_{2z} \end{bmatrix} = \begin{bmatrix} U_x & V_x & W_x \\ U_y & V_y & W_y \\ U_z & V_z & W_3 \end{bmatrix} \begin{bmatrix} (\dot{A}_2)_u \\ (\dot{A}_2)_v \\ (\dot{A}_2)_w \end{bmatrix} . \tag{37}$$

The elements of the transformation matrix included in the foregoing equation are the components of the unit vectors U, V and W, which are located along the axes of the G_2uvw coordinate system. The components of U along the main directions are obtained from:

$$[U_x, U_y, U_z]^T = \frac{1}{a_2} [A_{2x}, A_{2y}, A_{2z}]^T . \tag{38}$$

Since U and V are perpendicular to each other, the dot product (U . V) vanishes. This gives:

$$U_x V_x + U_y V_y + U_z V_z = 0 . \tag{39}$$

The components of V along the main directions are related to each other by:

$$V_x^2 + V_y^2 + V_z^2 = 1 . \tag{40}$$

V is parallel to the xy-plane, and therefore,

$$V_z = 0 . \tag{41}$$

By the use of eqs. (39) to (41), V is expressed as:

$$[V_x, V_y, V_z]^T = \frac{1}{h} [A_{2y}, - A_{2x}, 0]^T , \tag{42}$$

where: $$h = \sqrt{A_{2x}^2 + A_{2y}^2} . \tag{43}$$

W is defined by the cross product (U × V), and therefore it is given by:

$$[W_x, W_y, W_z]^T = \frac{1}{a_2 h} [A_{2x} A_{2z}, A_{2y} A_{2z}, - h^2]^T . \tag{44}$$

In eq. (37), when the components of $\dot{A}_2$ along the axes of the G_2uvw coordinate system are replaced by their expressions, which are given in eq. (36), and the components of U, V and W along the main directions are substituted by their expressions, which are presented in eqs. (38), (42) and (44), then this equation yields:

$$\Omega_v = \frac{\dot{A}_{2z}}{h} , \tag{45}$$

$$\Omega_w = \frac{A_{2y} \dot{A}_{2x} - A_{2x} \dot{A}_{2y}}{a_2 h} . \tag{46}$$

When the angular positions and velocities of the input and output links are known, the components of A_2 and $\dot{A}_2$ along the main directions can be determined from eqs. (21) and (34) respectively, and then Ω_v and Ω_w may be calculated by the use of equations (43), (45) and (46).

ACCELERATION

By differentiating eq. (20) with respect to time, the angular acceleration of the output link may be put in the form:

$$\ddot{\Gamma}_3 = \frac{D_7 + D_8 \cos \Gamma_3 + D_9 \sin \Gamma_3}{D_2 \sin \Gamma_3 - D_3 \cos \Gamma_3} , \tag{47}$$

where: $D_7 = 2a_1 \omega^2 (l_3 \sin \beta \cos \Gamma_1 - l_2 \sin \Gamma_1)$,

$$D_8 = 2a_1 a_3 \omega^2 \cos\beta \cos\theta_1 - \dot\theta_3 (D_2 \dot\theta_3 + 4\, a_1 a_3 \omega \cos\theta_1) \,, \tag{48}$$

$$D_9 = 2a_1 a_3 \omega^2 \sin\theta_1 - \dot\theta_3 (D_3 \dot\theta_3 + 4\, a_1 a_3 \omega \cos\beta \sin\theta_1) \,. \tag{49}$$

The accelerations of the mass centres G_1, G_2 and G_3 are expressed by double differentiation of eqs. (27), (29) and (28) with respect to time, as:

$$\begin{bmatrix} \ddot x_{g1} \\ \ddot y_{g1} \\ \ddot z_{g1} \end{bmatrix} = \omega^2 \begin{bmatrix} -u_{g1} \cos\theta_1 + v_{g1} \sin\theta_1 \\ -u_{g1} \sin\theta_1 - v_{g1} \cos\theta_1 \\ 0 \end{bmatrix} \tag{50}$$

$$\begin{bmatrix} \ddot x_{g2} \\ \ddot y_{g2} \\ \ddot z_{g2} \end{bmatrix} = \begin{bmatrix} (\mu-1)\,\omega^2 a_1 \cos\theta_1 - \mu a_3 \cos\beta (\ddot\theta_3 \sin\theta_3 + \dot\theta_3^2 \cos\theta_3) \\ (\mu-1)\,\omega^2 a_1 \sin\theta_1 + \mu a_3 (\ddot\theta_3 \cos\theta_3 - \dot\theta_3^2 \sin\theta_3) \\ \mu a_3 \sin\beta\, (\ddot\theta_3 \sin\theta_3 + \dot\theta_3^2 \cos\theta_3) \end{bmatrix} , \tag{51}$$

$$\begin{bmatrix} \ddot x'_{g3} \\ \ddot y'_{gz} \\ \ddot z_{gz} \end{bmatrix} = \begin{bmatrix} -u_{g3} (\ddot\theta_3 \sin\theta_3 + \dot\theta_3^2 \cos\theta_3) + v_{g3} (\dot\theta_3^2 \sin\theta_3 - \ddot\theta_3 \cos\theta_3) \\ u_{g3} (\ddot\theta_3 \cos\theta_3 - \dot\theta_3^2 \sin\theta_3) - v_{g3} (\ddot\theta_3 \sin\theta_3 + \dot\theta_3^2 \cos\theta_3) \\ 0 \end{bmatrix} . \tag{52}$$

DYNAMIC ANALYSIS

The angular momentum of link (1), after the addition of its balancing weight, with respect to point Q_1, is given by:

$$\begin{bmatrix} \bar H_{1u} \\ \bar H_{1v} \\ \bar H_{1w} \end{bmatrix} = \begin{bmatrix} I_{1u} & -I_{1uv} & -I_{1wu} \\ -I_{1uv} & I_{1v} & -I_{1vw} \\ -I_{1wu} & -I_{1vw} & I_{1w} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \omega \begin{bmatrix} -I_{1wu} \\ -I_{1vw} \\ I_{1w} \end{bmatrix} . \tag{53}$$

The matrix included in the foregoing equation is the inertia dyadic of the input crank, with its balancing weight included, taken with reference to its body-fixed coordinate system. The total moment of the inertia forces acting on this link with respect to point Q_1 is equal and opposite to the rate of change of its angular momentum with respect to the same reference point Q_1, and therefore it is given by:

$$\begin{bmatrix} \bar M_{1u} \\ \bar M_{1v} \\ \bar M_{1w} \end{bmatrix} = - \begin{bmatrix} \frac{d}{dt} (\bar H_{1u}) \\ \frac{d}{dt} (\bar H_{1v}) \\ \frac{d}{dt} (\bar H_{1w}) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} \bar H_{1u} \\ \bar H_{1v} \\ \bar H_{1w} \end{bmatrix} , \tag{54}$$

where $d(\bar H_{1u})/dt$, $d(\bar H_{1v})/dt$ and $d(\bar H_{1w})/dt$ are the time derivatives of the magnitudes of the components $\bar H_{1u}$, $\bar H_{1v}$ and $\bar H_{1w}$ respectively. When the origin O is taken a reference point, the resultant moment of these inertia forces is obtained from:

$$\begin{bmatrix} M_{1x} \\ M_{1y} \\ M_{1z} \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar M_{1u} \\ \bar M_{1v} \\ \bar M_{1w} \end{bmatrix} - m_1 \begin{bmatrix} x_{q1} \\ y_{q1} \\ z_{q1} \end{bmatrix} \times \begin{bmatrix} \ddot x_{g1} \\ \ddot y_{g1} \\ \ddot z_{g1} \end{bmatrix} . \tag{55}$$

By the use of eqs. (15), (50) and (53) to (55), the components of M_1 along the axes of the Oxyz coordinate system, is expressed as:

$$\begin{bmatrix} M_{1x} \\ M_{1y} \\ M_{1z} \end{bmatrix} = \begin{bmatrix} -I_{1wu}\,\omega^2 \sin\Gamma_1 - I_{1vw}\,\omega^2 \cos\Gamma_1 + m_1 l_1 \ddot{y}_{g1} \\ I_{1wu}\,\omega^2 \cos\Gamma_1 - I_{1vw}\,\omega^2 \sin\Gamma_1 - m_1 l_1 \ddot{x}_{g1} \\ m_1 l_2 \ddot{x}_{g1} \end{bmatrix}. \quad (56)$$

Similarly, the components of M_3, which is the resultant moment of the inertia forces acting on link (3), after the addition of its balancing weight, with respect to the origin O, along the axes of the Ox'y'z' coordinate system, is expressed as:

$$\begin{bmatrix} M_{3x'} \\ M_{3y'} \\ M_{3z'} \end{bmatrix} = \begin{bmatrix} I_{3wu}(\ddot{\Gamma}_3\cos\Gamma_3 - \dot{\Gamma}_3^2\sin\Gamma_3) - I_{3vw}(\ddot{\Gamma}_3\sin\Gamma_3 + \dot{\Gamma}_3^2\cos\Gamma_3) + m_3 l_3 \ddot{y}'_{g3} \\ I_{3wu}(\ddot{\Gamma}_3\sin\Gamma_3 + \dot{\Gamma}_3^2\cos\Gamma_3) + I_{3vw}(\ddot{\Gamma}_3\cos\Gamma_3 - \dot{\Gamma}_3^2\sin\Gamma_3) - m_3 l_3 \ddot{x}'_{g3} \\ -I_{3w}\,\ddot{\Gamma}_3 \end{bmatrix}. \quad (57)$$

The later components are related to the components of M_3 along the main directions by:

$$\begin{bmatrix} M_{3x} \\ M_{3y} \\ M_{3z} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} M_{3x'} \\ M_{3y'} \\ M_{3z'} \end{bmatrix}. \quad (58)$$

Eqs. (57) and (58) yield :

$$\begin{bmatrix} M_{3x} \\ M_{3y} \\ M_{3z} \end{bmatrix} = \begin{bmatrix} m_3 l_3 \ddot{y}'_{g3}\cos\beta - I_{3w}\ddot{\Gamma}_3\sin\beta + I_{3wu}\cos\beta(\ddot{\Gamma}_3\cos\Gamma_3 - \dot{\Gamma}_3^2\sin\Gamma_3) \\ m_3 l_3 \ddot{x}'_{g3} + I_{3wu}(\ddot{\Gamma}_3\sin\Gamma_3 + \dot{\Gamma}_3^2\cos\Gamma_3) \\ -m_3 l_3 \ddot{y}'_{g3}\sin\beta - I_{3w}\ddot{\Gamma}_3\cos\beta + I_{3wu}\sin\beta(\dot{\Gamma}_3^2\sin\Gamma_3 - \ddot{\Gamma}_3\cos\Gamma_3) \end{bmatrix}$$
$$\begin{matrix} - I_{3vw}\cos\beta(\ddot{\Gamma}_3\sin\Gamma_3 + \dot{\Gamma}_3^2\cos\Gamma_3) \\ + I_{3vw}(\ddot{\Gamma}_3\cos\Gamma_3 - \dot{\Gamma}_3^2\sin\Gamma_3) \\ + I_{3vw}\sin\beta(\ddot{\Gamma}_3\sin\Gamma_3 + \dot{\Gamma}_3^2\cos\Gamma_3) \end{matrix} \quad . \quad (59)$$

The coupler is a straight rod with a small cross-section, and therefore all its moments and products of inertia with respect to its body-fixed coordinate system may be ignored, except its moments of inertia about the axes v and w. Hence, its angular momentum, with reference to its mass centre G_2 is given by:

$$[\bar{H}_{2u}, \bar{H}_{2v}, \bar{H}_{2w}]^T = I\,[0, \Omega_v, \Omega_w]^T , \quad (60)$$

where $I = I_v = I_w$. The angular momentum of the coupler with respect to the origin O is obtained from:

$$\begin{bmatrix} H_{2x} \\ H_{2y} \\ H_{2z} \end{bmatrix} = \begin{bmatrix} U_x & V_x & W_x \\ U_y & V_y & W_y \\ U_z & V_z & W_z \end{bmatrix} \begin{bmatrix} \bar{H}_{2u} \\ \bar{H}_{2v} \\ \bar{H}_{2w} \end{bmatrix} + m_2 \begin{bmatrix} x_{g2} \\ y_{g2} \\ z_{g2} \end{bmatrix} \times \begin{bmatrix} \dot{x}_{g1} \\ \dot{y}_{g1} \\ \dot{z}_{g1} \end{bmatrix}. \quad (61)$$

Eqs. (60) and (61) yields:

$$\begin{bmatrix} H_{2x} \\ H_{2y} \\ H_{2z} \end{bmatrix} = I \begin{bmatrix} V_x\Omega_v + W_x\Omega_w \\ V_y\Omega_v + W_y\Omega_w \\ V_z\Omega_v + W_z\Omega_w \end{bmatrix} + m_2 \begin{bmatrix} y_{g2}\dot{z}_{g2} - \dot{y}_{g2} z_{g2} \\ z_{g2}\dot{x}_{g2} - \dot{z}_{g2} x_{g2} \\ x_{g2}\dot{y}_{g2} - \dot{x}_{g2} y_{g2} \end{bmatrix}. \quad (62)$$

The resultant moment of the inertia forces acting on the coupler, with respect to the origin O, is expressed, by differentiating the components of H_2 in the foregoing equation with respect to time, as:

$$\begin{bmatrix} M_{2x} \\ M_{2y} \\ M_{2z} \end{bmatrix} = -I \begin{bmatrix} V_x \alpha_v + \dot{V}_x \Omega_v + W_x \alpha_w + \dot{W}_x \Omega_w \\ V_y \alpha_v + \dot{V}_y \Omega_v + W_y \alpha_w + \dot{W}_y \Omega_w \\ V_z \alpha_v + \dot{V}_z \Omega_v + W_z \alpha_w + \dot{W}_z \Omega_w \end{bmatrix} - m_2 \begin{bmatrix} y_{g2}\, \ddot{z}_{g2} - \ddot{y}_{g2}\, z_{g2} \\ z_{g2}\, \ddot{x}_{g2} - \ddot{z}_{g2}\, x_{g2} \\ x_{g2}\, \ddot{y}_{g2} - \ddot{x}_{g2}\, y_{g2} \end{bmatrix}, \tag{63}$$

where α_v and α_w are the time derivatives of the magnitudes of the components Ω_v and Ω_w respectively. These derivatives, as well as the components of $\dot{V}$ and $\dot{W}$, are expressed, by the use of eqs. (38) and (42) to (46), as:

$$\dot{V}_x = (h^2 \dot{A}_{2y} + A_{2y} A_{2z} \dot{A}_{2z})/h^3 , \tag{64}$$

$$\dot{V}_y = -(h^2 \dot{A}_{2x} + A_{2x} A_{2z} \dot{A}_{2z})/h^3 , \tag{65}$$

$$\dot{V}_z = 0 , \tag{66}$$

$$\dot{W}_x = (a_2^2 A_{2x} \dot{A}_{2z} + h^2 A_{2z} \dot{A}_{2x})/a_2 h^3 , \tag{67}$$

$$\dot{W}_y = (a_2^2 A_{2y} \dot{A}_{2z} + h^2 A_{2z} \dot{A}_{2y})/a_2 h^3 , \tag{68}$$

$$\dot{W}_z = A_{2z} \dot{A}_{2z}/a_2 h , \tag{69}$$

$$\alpha_v = (h^2 \ddot{A}_{2z} + A_{2z} \dot{A}_{2z}^2)/h^3, \tag{70}$$

$$\alpha_w = [(h^2(A_{2y} \ddot{A}_{2x} - A_{2x} \ddot{A}_{2y}) + A_{2z} \dot{A}_{2z}(A_{2y} \dot{A}_{2x} - A_{2x} \dot{A}_{2y})]/a_2 h^3 . \tag{71}$$

In order to find the components of $\ddot{A}_2$, which are included in eqs. (70) and (71), eq. (34) is differentiated with respect to time.
This gives:

$$\begin{bmatrix} \ddot{A}_{2x} \\ \ddot{A}_{2y} \\ \ddot{A}_{2z} \end{bmatrix} = \begin{bmatrix} a_1 \omega^2 \cos\Gamma_1 - a_3 \cos\beta(\dot{\Gamma}_3^2 \cos\Gamma_3 + \ddot{\Gamma}_3 \sin\Gamma_3) \\ a_1 \omega^2 \sin\Gamma_1 + a_3(\ddot{\Gamma}_3 \cos\Gamma_3 - \dot{\Gamma}_3^2 \sin\Gamma_3) \\ a_3 \sin\beta(\dot{\Gamma}_3^2 \cos\Gamma_3 + \ddot{\Gamma}_3 \sin\Gamma_3) \end{bmatrix} . \tag{72}$$

DERIVATION OF THE SHAKING MOMENT EXPRESSION

The shaking moment, with respect to a reference point, is defined as the resultant moment, with respect to the reference point, transmitted to ground due to the inertia effects of the moving links. This moment may be obtained by summing the moments of the inertia forces acting on the moving links about the reference point, since the reaction forces and couples at all joints appear in pairs of equal magnitudes and opposite directions. Force-balancing of the mechanism under consideration implies that the inertia forces acting on the moving links are balanced or reduced to a pure couple, and therefore the shaking moment is invariant of the reference point. In the present work, the expression defining the shaking moment is derived by summing the moments of the inertia forces acting on links (1), (2) and (3), with respect to the origin O. As mentioned above, the balancing weights attached to links (1) and (3) are designed to yield the minimum shaking moment, but without changing their masses or the positions of their mass centres, which have been determined by the force-balancing conditions. Hence, the shaking moment is expressed in terms the unknown inertia parameters of links (1) and (3), with their balancing weights, which are included in eqs. (56) and (59). In order to simplify the resulting expression, these parameters are denoted as follows:

$J_1 = I_1$, $J_2 = I_1$

When the components of M_1 and M_3 are substituted by the expressions given in eqs. (56) and (59), the components of the shaking moment along the main directions may be put in the form:

$$M_x = \sum_{i=0}^{4} f_{xi} J_i \ , \qquad (73)$$

$$M_y = \sum_{i=0}^{4} f_{yi} J_i \ , \qquad (74)$$

$$M_z = \sum_{i=0}^{4} f_{zi} J_i \ , \qquad (75)$$

where:

$$J_0 = 1 \ , \qquad (76)$$

$$f_{x0} = -m_1 l_2 \ddot{z}_{g1} + m_1 l_1 \ddot{y}_{g1} + M_{2x} + m_3 l_3 \ddot{y}'_{g3} \cos\beta - I_{3w} \ddot{\Gamma}_3 \sin\beta \ , \qquad (77)$$

$$f_{y0} = -m_1 l_1 \ddot{x}_{g1} + M_{2y} - m_3 l_3 \ddot{x}'_{g3} \ , \qquad (78)$$

$$f_{z0} = m_1 l_2 \ddot{x}_{g1} + M_{2z} - m_3 l_3 \ddot{y}'_{g3} \sin\beta - I_{3w} \ddot{\Gamma}_3 \cos\beta \ , \qquad (79)$$

$$f_{x1} = f_{y2} = -\omega^2 \sin\Gamma_1 \ , \qquad (80)$$

$$f_{y1} = -f_{x2} = \omega^2 \cos\Gamma_1 \ , \qquad (81)$$

$$f_{z1} = f_{z2} = 0 \ , \qquad (82)$$

$$f_{y3} = \ddot{\Gamma}_3 \sin\Gamma_3 + \dot{\Gamma}_3^2 \cos\Gamma_3 \ , \qquad (83)$$

$$f_{y4} = \ddot{\Gamma}_3 \cos\Gamma_3 - \dot{\Gamma}_3^2 \sin\Gamma_3 \ , \qquad (84)$$

$$f_{x3} = f_{y4} \cos\beta \ , \qquad (85)$$

$$f_{z3} = -f_{y4} \sin\beta \ , \qquad (86)$$

$$f_{x4} = -f_{y3} \cos\beta \ , \qquad (87)$$

$$f_{z4} = f_{y3} \sin\beta \ . \qquad (88)$$

When the angular position of the input crank is known, the f coefficients, which are defined in the above equations, can be calculated by the following procedure.

1. Calculate Γ_3, $\dot{\Gamma}_3$ and $\ddot{\Gamma}_3$ from eqs. (23) to (26), (30) to (33) and (47) to (49).
2. Calculate the coordinates of G_2 from eq. (29).
3. Calculate the components of the accelerations of G_1, G_2 and G_3 from eqs. (50) to (52).
4. Calculate the components of A_2, $\dot{A}_2$ and $\ddot{A}_2$ from eqs.(21),(34) and (72).
5. Calculate h from eq. (43).
6. Calculate the components of V, W and Ω from eqs. (42) and (44) to (46).
7. Calculate the components of $\dot{V}$, $\dot{W}$, and α from eqs. (64) to (71).
8. Calculate the components of M_2 from eq. (63).
9. Calculate the f coefficients from eqs. (77) to (88).

Thus, eqs. (73) to (75) define the components of the shaking moment, at a known input crank position, as linear functions of the unknown products of inertia J_1 to J_4.

MINIMIZATION OF THE RMS SHAKING MOMENT

The magnitude of the shaking moment is given by:

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2} \ . \qquad (89)$$

Substitution by eqs. (73) to (75) yields :

$$M^2 = \sum_{i=0}^{4} \sum_{j=0}^{4} (f_{xi} f_{xj} + f_{yi} f_{yj} + f_{zi} f_{zj}) J_i J_j \,. \tag{90}$$

The RMS shaking moment M_r is obtained from:

$$M_r^2 = \frac{1}{2\pi} \int_0^{2\pi} M^2 \, d\psi_1 \,. \tag{91}$$

By the use of eq. (90), M_r^2 can be written in the form:

$$M_r^2 = \sum_{i=0}^{4} \sum_{j=0}^{4} F_{ij} J_i J_j \,, \tag{92}$$

where:
$$F_{ij} = \frac{1}{2\pi} \int_0^{2\pi} (f_{xi} f_{xj} + f_{yi} f_{yj} + f_{zi} f_{zj}) \, d\psi_1 \,. \tag{93}$$

Eq. (92) gives M_r^2 as a function of the unknowns J_1 to J_4. In order to find the values of these unknowns which produce the minimum shaking moment, eq. (92) is differentiated partially with respect to each of the unknowns and the derivatives are put equal to zero. This gives:

$$F_{i1} J_1 + F_{i2} J_2 + F_{i3} J_3 + F_{i4} J_4 = -F_{i0} \qquad (i=1,2,3,4). \tag{94}$$

Double partial differentiation of M_r^2 with respect to each of the unknowns yields:

$$\frac{\partial^2 M_r^2}{\partial J_i^2} = 2 F_{ii} \qquad (i=1,2,3,4). \tag{95}$$

From eqs. (93) and (95), it can be seen that $\partial^2 M_r^2/\partial J_i^2$ is always nonnegative, and therefore the values of the unknowns J_1 to J_4, which are obtained by solving the four equations represented by eq. (94), correspond to the minimum shaking moment.

DESIGN OF THE BALANCING WEIGHTS

After finding the products of inertia I_{3wu} and I_{3vw}, which refer to link (3) with its balancing, the corresponding products of inertia I_{b3wu} and I_{b3vw}, which belong to the balancing weight only, are determined. The later values are employed in calculating the angles α and γ, which are necessary for specifying the balancing weight. Referring to Fig. 2, B_3 is the mass centre of the balancing weight, and $B_3u''v''w''$ is an auxiliary coordinate system fixed to link (3) and located such that, the w''-axis coincides with the axis of the balancing weight and the u''-axis lies in the plane containing the major axes of the elliptical plane sides of the balancing weight. $B_3u'v'w'$ is another auxiliary coordinate system fixed to link (3). In this coordinate system the w'-axis coincides with the w''-axis, whereas the u'-axis is parallel to the u-axis. I_{b3wu} and I_{b3vw} are related to the inertia products of the balancing weight with reference to the $B_3u'v'w'$ coordinate system by:

$$I_{b3wu} = I_{b3w'u'} + m_{b3} w_{b3} u_{b3} \,, \tag{96}$$

$$I_{b3vw} = I_{b3v'w'} + m_{b3} v_{b3} w_{b3} \,. \tag{97}$$

The later inertia products can be expressed in terms of the inertia products of the balancing weight with respect to the $B_3u''v''w''$ coordinate system as:

$$I_{b3w'u'} = I_{b3w''u''} \cos\gamma - I_{b3w''v''} \sin\gamma \,, \tag{98}$$

The inertia products $I_{3bw''u''}$ and $I_{3bv''w''}$ are given by:

$$I_{b3w''u''} = -\frac{m_{b3}\, r^2}{4\tan\alpha}, \tag{100}$$

$$I_{b3v''w''} = 0 . \tag{101}$$

From eqs. (96) to (101):

$$\alpha = \tan^{-1} \frac{m_{b3}\, r^2}{4\sqrt{(I_{b3wu}-m_{b3}\, w_{b3}\, u_{b3})^2+(I_{b3vw}-m_{b3}\, v_{b3}\, w_{b3})^2}}, \tag{102}$$

$$\gamma = \tan^{-1} \frac{I_{b3vw}-m_{b3}\, v_{b3}\, w_{b3}}{I_{b3wu}-m_{b3}\, w_{b3}\, u_{b3}} . \tag{103}$$

Knowing the angles α and γ , the balancing weight attached to link (3) is specified. The balanchig weight fixed to link (1) may be designed in a similar way, or in any form which gives I_{1wu} and I_{1vw} the values obtained by the solution of eq. (94).

EXAMPLE

The proposed method was applied to an RGGR mechanism with the following dimensionless parameters.

$l_1/a_1 = 5$	$l_2/a_1 = 0.5$	$l_3/a_1 = 5$
$\beta = 80°$	$a_2/a_1 = 6$	$a_3/a_1 = 4$
$u_{g1}/a_1 = 0.5$	$v_{g1} = w_{g1} = 0$	$I_{1wu} = I_{1vw} = 0$
$m_2/m_{o1} = 3$	$c_2/a_1 = 3$	$I/m_{o1}a_1^2 = 9$
$m_{o3}/m_{o1} = 4$	$u_{g3}/a_1 = 2$	$v_{g3} = w_{g3} = 0$
$I_{o3wu} = I_{o3vw} = 0$	$I_{o3w}/m_{o1}\, a_1^2 = 22$.	

The force-balancing conditions were applied with $m_{b1}/m_{o1} = 2$ and $\int L = m_{o1}/a_1^2$. This gave:

$u_{b1}/a_1 = -1$	$v_{b1} = w_{b1} = 0$.	
$m_{b3}/m_{o1} = 8.5$	$u_{b3}/a_1 = -1.645$	$v_{b3} = w_{b3} = 0$
$r/a_1 = 1.645$	$I_{b3w}/(m_{o1}\, a_1^2) = 34.56$	

The shaking moment optimization yielded:

$I_{1wu}/(m_{o1}\, a_1^2) = -59.94$	$I_{1vw}/(m_{o1}\, a_1^2) = -55.75$
$\alpha = 28.5°$	$\gamma = 226.6°$

By the addition of these balancing weights, the shaking force was completly balanced, and RMS shaking moment was reduced from 22.38 $m_{o1}\,\omega^2\, a_1^2$ to 14.25 $m_{o1}\,\omega^2\, a_1^2$, i.e. to 64% of its initial value.

CONCLUSIONS

The following conclusions may be drawn regarding the force-balancing and shaking moment optimization of an RGGR mechanism with arbitrary link masses and dimensions, except its coupler which is assumed to be a thin straight rod .

1. The shaking force can be balanced completely by attaching two balancing weights to the input and output links.

weights, and therefore more conditions are necessary for specifying the balancing weights completely.

3. Complete balancing of the shaking force is independent of the input speed, since the total mass centre of the moving links remains stationary during the mechanism motion.

4. When the input crank is rotating at a uniform speed, its moment of inertia about its axis of rotation has no influence on the resulting shaking moment, and the optimization of the shaking moment defines its inertia products I_{b1wu} and I_{b1vw} only.

5. The output link, is assumed to be moving at a variable speed, and therefore it is taken as a cylinder located such that one of the generators of its cylindrical surface coincides with the axis of rotation of the output link. This condition together with the conditions of the shaking moment optimization are used for specifying this balancing weight completely. The inclination of the parallel plane sides of this balancing weight allows for adjusting its inertia products I_{b3wu} and I_{b3vw}, to minimize the RMS shaking moment, without affecting its inertia moment I_{b3w}.

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