A.M.E.

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SINGULAR PERTURBATIONS AND AIRCRAFT LONGITUDINAL DYNAMICS

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ABSTRACT

The singular perturbations method is used to find a simplified solution of the longitudinal aircraft dynamics. The variables are devided into slow and fast. Then the steady state fast variables are used as input to the slow subsystem. The obtained simplified solution of the short period mode is the same as previously known solution of this mode. The obtained solution of the phugoid mode is basically different from the well known approximate solution of this mode. Calculations proved that the obtained approximate solution is very near to the exact one.

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NOMENCLATURE

C_L	lift coefficient
C_m	pitching moment coefficient
C_x	drag coefficient
C_z	force coefficient in the direction of z axis
D	differential operator
i	nondimensional moment of inertia around y axis
K_n	static stability margin
q	perturbation pitching velocity
u	perturbation velocity in x axis direction
α	perturbation angle of attack
θ	perturbation pitch angle
μ	nondimensional mass of the airplane

SUBSCRIPTS

o	reference steady state
q	derivative of the coefficient with respect to q
u	derivative of the coefficient with respect to u
α	derivative of the coefficient with respect to α
$\dot{\alpha}$	derivative of the coefficient with respect to $\dot{\alpha}$

INTRODUCTION

Singular perturbations method is recently used in many works concerning order reduction, separation of time scales and control system analysis and design. In Ref [1] an excellent review of the method of singular perturbations and its applications in different areas is given. The theoritical bases of the method could be found in Ref [2]. In Ref [3] the method is used to find the optimum flight path angle using separate time scales. Here the method of singular perturbations will be used to reduce the order of the dynamic system. The application will be directed towards getting a new simplified solution of the phugoid mode of an airplane.

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ORDER REDUCTION

The order of any dynamic system could be reduced if the state vector is composed from a group of slow variables and another one of fast variables. The order reduction could be performed if the following assumptions are satisfied:

a) The frequency of the fast variables must be at least ten times higher than that of the slow variables.

b) At least the fast variables are stable.

Assuming that the system model has the following form:

$$\dot{x} = Ax + By \tag{1}$$

$$\varepsilon \dot{y} = Cx + Dy \tag{2}$$

where A,B,C and D are matrices of proper dimensions, x is an n dimensions vector, y is an m dimensions vector, ε is a small number. The order of the system is m + n.Equations (1) and (2) represent a linearized model of a dynamic system. It is clear that the vector x represents the slow variables and the vector y represents the fast variables. In order to apply the singular perturbation method the system must be cast in the above given form. If the system is exited then at the begining the slow variables will not change and the system will be gevorned by

$$\varepsilon \dot{y} = Dy \tag{3}$$

The corresponding modes are stable if the real part of the m eigenvalues of the matrix D are negative. The steady state of the fast variables is obtained from equation (2) putting $\varepsilon = 0$ and solving for y

$$y = -D^{-1}C\ x$$

The steady state of the fast variables are used as input to the slow variables. Substituting into equation (1) we get

$$\dot{x} = [A - BD^{-1}C]\ x \tag{4}$$

So the original system of order n + m is reduced to two subsystems. One is a fast subsystem and its eigenvalues are those of the matrix D. The other is a slow subsystem and its eigenvalues are those of the matrix $[A - BD^{-1}C]$ The system is stable if the n eigenvalues of the matrix $[A - BD^{-1}C]$ as well as the m eigenvalues of the matrix D have negative real part.

MATHEMATICAL MODEL

The equations of longitudinal motion in nondimensional linearized form are according to Ref [4]:

$$(2\mu D - C_{xu})\,u - C_{x\alpha}\,\alpha + C_{L_0}\,\theta = 0 \qquad (5.a)$$

$$(2C_{L_0} - C_{zu})\,u + (2\mu D - C_{z\dot\alpha} D - C_{z\alpha})\,\alpha - (2\mu + C_{zq})\,D\theta = 0 \qquad (5.b)$$

$$-C_{mu}\,u - (C_{m\dot\alpha} D + C_{m\alpha})\,\alpha + (i D^2 - C_{mq} D)\,\theta = 0 \qquad (5.c)$$

The classical phugoid approximation uses first and second equation and drops the variable α (see Ref (4)). So the phugoid mode characteristic equation is

$$\lambda^2 - \frac{C_{xu}}{2\mu}\lambda + \frac{C_{L_0}^2}{2\mu^2} = 0 \qquad (6)$$

In order to use the method of singular perturbations, equation (5) must be formulated in a matrix form and the variables must be devided into slow and fast. From the will known solutions of the longitudinal aircraft dynamics it is possible to say that the slow variables are u and θ while the fast variables are α and q. Arranging the equations of motion in matrix form the following form could be easily found:

$$\begin{bmatrix} \dot u \\ \dot\theta \\ \dot\alpha \\ \dot q \end{bmatrix} = \begin{bmatrix} \frac{C_{xu}}{2\mu} & -\frac{C_{L_0}}{2\mu} & \frac{C_{x\alpha}}{2\mu} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{C_{L_0}}{\mu} & 0 & \frac{C_{z\alpha}}{2\mu} & 1 \\ -\frac{C_{m\dot\alpha} C_{L_0}}{\mu\, i\,[illegible]} & 0 & \frac{C_{m\alpha}}{[illegible]} + \frac{C_{z\alpha} C_{m\dot\alpha}}{[illegible]} & \frac{C_{m\dot\alpha} + C_{mq}}{[illegible]} \end{bmatrix} \begin{bmatrix} u \\ \theta \\ \alpha \\ q \end{bmatrix} \qquad (7)$$

It is easy to see that the approximate solution of the short period mode is the same. The phugoid approximate solution is obtained as a dominant mode of the above system. The steady state values of the fast variables are

$$\alpha = 2\left[C_{Lo} C_{mq} / (C_{z\alpha} C_{mq} - 2\mu C_{m\alpha})\right] u \qquad (8.a)$$

$$q = -2\left[C_{Lo} C_{m\alpha} / (C_{z\alpha} C_{mq} - 2\mu C_{m\alpha})\right] u \qquad (8.b)$$

It is seen from the above equation that the steady state fast variables α and q are function of the perturbation velocity u only and are independant on the perturbation elevation angle θ . It is quite acceptable result from the physical point of view. Substituting into equation (7) the reduced system representing the dominant mode is obtained as follows:

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{C_{xu}}{2\mu} + \dfrac{C_{x\alpha} C_{Lo} C_{mq}}{\mu(C_{z\alpha} C_{mq} - 2\mu C_{m\alpha})} & -\dfrac{C_{Lo}}{2\mu} \\ -\dfrac{2 C_{Lo} C_{m\alpha}}{C_{z\alpha} C_{mq} - 2\mu C_{m\alpha}} & 0 \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} \qquad (9)$$

The above equation is a new representation of the simplified phugoid mode. The characteristic equation could be easily found, after some manipolation it is possible to write it as:

$$\lambda^2 - \left[\frac{C_{xu}}{2\mu} + \frac{C_{x\alpha} C_{Lo} C_{mq}}{\mu(C_{z\alpha} C_{mq} - 2\mu C_{m\alpha})}\right]\lambda - \frac{C_{Lo}^2 C_{m\alpha}}{\mu(C_{z\alpha} C_{mq} - 2\mu C_{mq})} = 0 \qquad (10)$$

Comparing equation (10) with equation (6) it is possible to see how far they are different. The obtained simplified phugoid solution is dependant on the static stability $C_{m\alpha}$, damping in pitch C_{m_q} and slope of lift curve $C_{z\alpha}$. These derivatives do not appear in the classical formula of the simplified solution. As expected the solution is independant upon the moment of inertia of the airplane which is in agreament with the known simpllified solution.

RESULTS AND DISCUSSIONS

The method is applied on an airplane, its data is given in Ref [4]. The exact solution is compared to both classical approximate solution given by equation (6) and the new solution given by equation (10).
As mentioned previously the short period solution using the introduced method gives the same result as the well known approximate solution of the short period mode. So the discussions and analysis will be limited to the phugiod mode.
In order to analyse the obtained solution and to compare it with the exact solution, the static margin K_n was chosen as a parameter. The period T and the damping ($N_{\frac{1}{2}}$) are found as function of the static margin K_n. The period of the phugoid mode obtained by this method is nearly the same as the exact solution. This is seen in Fig(1) where the curves of both solution coinside. The classical approximate solution gives constant value for the period. This means that the introduced method gives an excellent approximation for the period of the phugoid mode.
The damping which is expressed by the number of cycles to half amplitude $N_{\frac{1}{2}}$ is seen in Fig(2). It is clear that the solution gives better approximation for the damping than the classical approximate solution. The dependence of $N_{\frac{1}{2}}$ upon the static margin K_n has the same behaviour as that of the exact solution. At high values of K_n the approximation is very good. As the value of K_n decreases the difference between the exact and the introduced solution increases, but it gives better results than the well known approximate solution of the phugoid mode. In the region of negative static stability (K_n negative) the results are far from the exact. This is expected, since the mothod is based on the assumption that there are two modes one fast and the other is slow, and the fast is a stable one . But for negative static stability the short period mode is a divergant one. Hence the phugoid mode obtained by the presented method is not real.
The root locus plot is given in fig(5). It is seen that the phugoid

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branch corresponding to statically stable airplane (K_n positive) is the same for both exact and presented solution. For small negative values of K_n the root locus plots are different. As K_n increases in the negative direction the root locus plots become closer till they coiside for $K_n < -0.1$. The method is applied to a jet trainer and the results are given in Fig(3) and Fig(4). They give similar results to those shown in fig.(1) and fig.(2).

CONCLUSION

A new solution of the approximate phugoid mode is obtained. The obtained solution gives excellent approximation for the period of this mode. The damping is very near to that of the exact solution. The method proves to be effective in the studied case. Also the method could be used to introduce the effect of the elastic mode on the longitudinal dynamics following the same procedure of this work.

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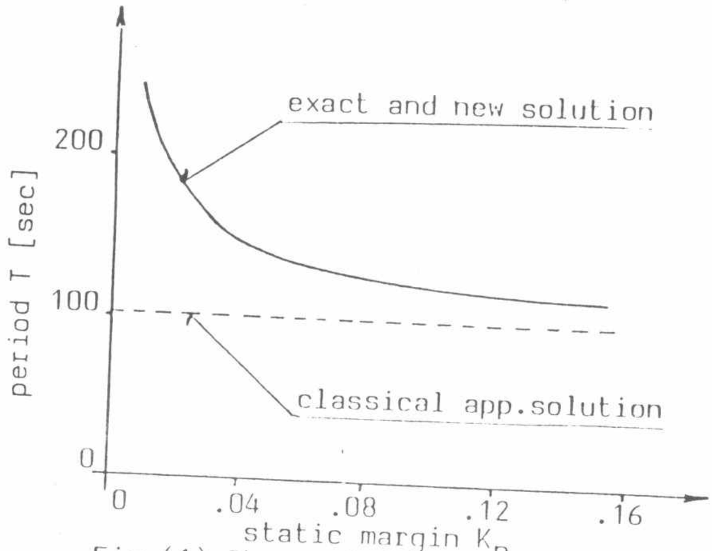

Fig.(1),The period as function of static margin(jet transport)

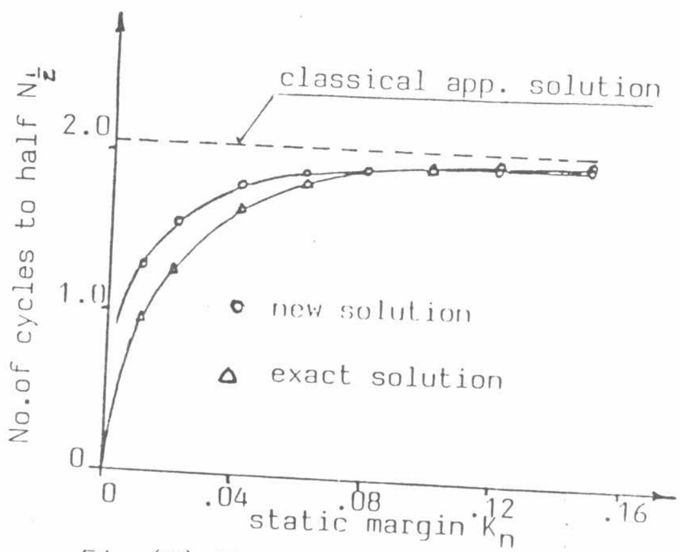

Fig.(2),Thedamping as function of static margin(jet transport)

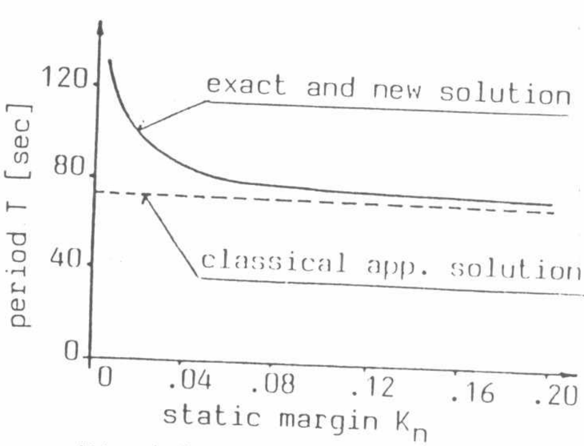

Fig.(3),The period as function of static margin (jet trainer)

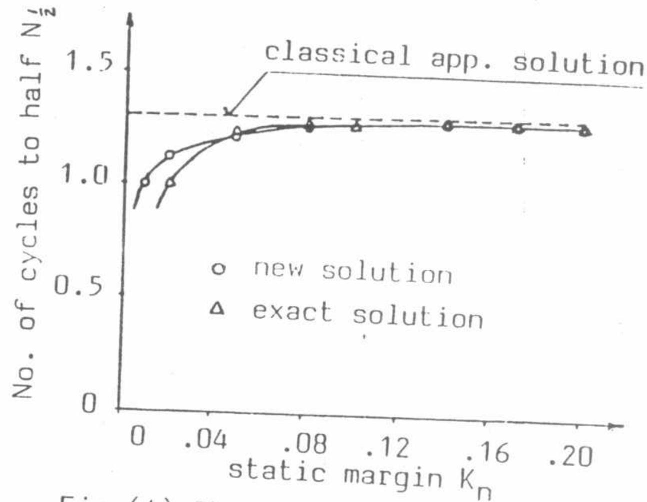

Fig.(4),The damping as function of static margin(jet trainer)

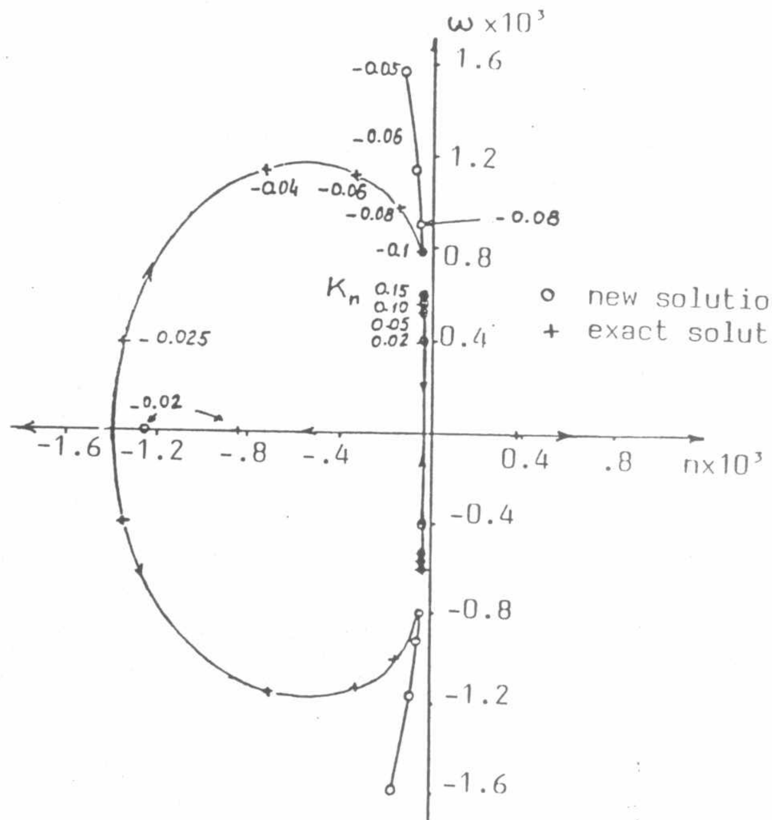

Fig.(5),Root locus plot for different values of static margin(jet transport)