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MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

1 ON PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL

2 EQUATIONS

3 By

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4 ABSTRACT

In this paper we establish sufficient conditions for the existence of periodic solutions of the equation

$$\ddot{x} + K(x - a)\ (x - b)\ \dot{x} + \epsilon\, x^{2n+1}\ (x - c) = 0,$$

This equation does not satisfy the condition $xg(x) > 0$ for $|x| > 0$ which was assumed in [3] and [5] .

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INTRODUCTION

In this paper we are going to establish sufficient conditions to be satisfied by the constants k,a,b, c and ϵ to prove the existence of a nonconstant periodic solution of the nonlinear second order differential equation

$$\ddot{x} + K(x-a)(x-b)\dot{x} + \epsilon x^{2n+1}(x-c) = 0 \qquad (1)$$

The equation is a form of lineard's equation

$$\ddot{x}+f(x)\dot{x}+g(x)= 0 \qquad (2)$$

which discribes many physical phenomenas.The equation does not satisfy the condition which was commonly used by [3], [5], and others.To prove the existence of a periodic solutions of (1) we suppose that $K > 0, \epsilon > 0$, a , b and c are real constants. Equation(1) is equivalent to the system,

$$\dot{x} = y, \ \dot{y} =-f(x)y- g(x) \qquad (3)$$

where $f(x) =k(x-a)(x-b)$ (4)

$g(x)=\epsilon x^{2n+1}(x- c)$ (5)

To prove our result we shall use a technique similar to that used in [3] and [5] to investigate the phase digrame of the system (3).

MAIN RESULT

Theorem

Suppose that

i) $c < a < 0 < b$

ii) there are two numbers x_1, w ; $c < x_1 < a$; $w > 1$

such that

$$2w\left(1+\frac{L}{\sqrt{2N}}\right)L \leq B(x_1, a)$$

iii) $N^2 \leq \epsilon w^2 \int_{x_1}^{c_1} x^{2n+1} (x-c)\, dx$

iv) $\lim_{x \to \infty} G(x) > \sqrt{2}\, N + L;$

(v) where
$$\begin{cases} B(x_1, a) = k\left[\dfrac{(x_1-a)^3}{6} - \dfrac{(x_1-a)^2 (x_1-b)}{2}\right] \\ L = \dfrac{G}{N} + K\dfrac{(b-a)^3}{6}, \\ G = \max\{G(a), G(b)\} \\ N = \left[\epsilon \int_a^c x^{2n+1} (x-c)\, dx\right]^{\frac{1}{2}} \end{cases}$$

Then equation(1) has at least one nonconstant periodic solution.

<u>Proof</u>

Consider instead of equation(1) the equivalent system (3) .The only two critical points of the system (3) are $0=(0,0)$ which is unstable and $A_o=(c,0)$ which is a saddle point.With A_o is associated four separatrices T_+ T_-, U_+ and U_-.

For t increasing,the separatrix T_+ leaves A_o and enters region:

$$V = \left\{(x, y) \mid c < x < a,\ \frac{-\epsilon x^{2n+1} (x-c)}{k(x-a)(x-b)} > y > 0\right\},$$

Now ,we shall prove that T_+ is a contracting spiral .We have different cases.

case 1:Let us suppose that T_+ intersects the line $x=b$ and let $P_1=(a,y_1)$, $P_2=(0,y_2)$ and $P_3=(b,y_3)$, denote the intersections of T_+ with the lines $x=a$, $x=0$, and $x=b$ respectively as t increases (Fig.1)

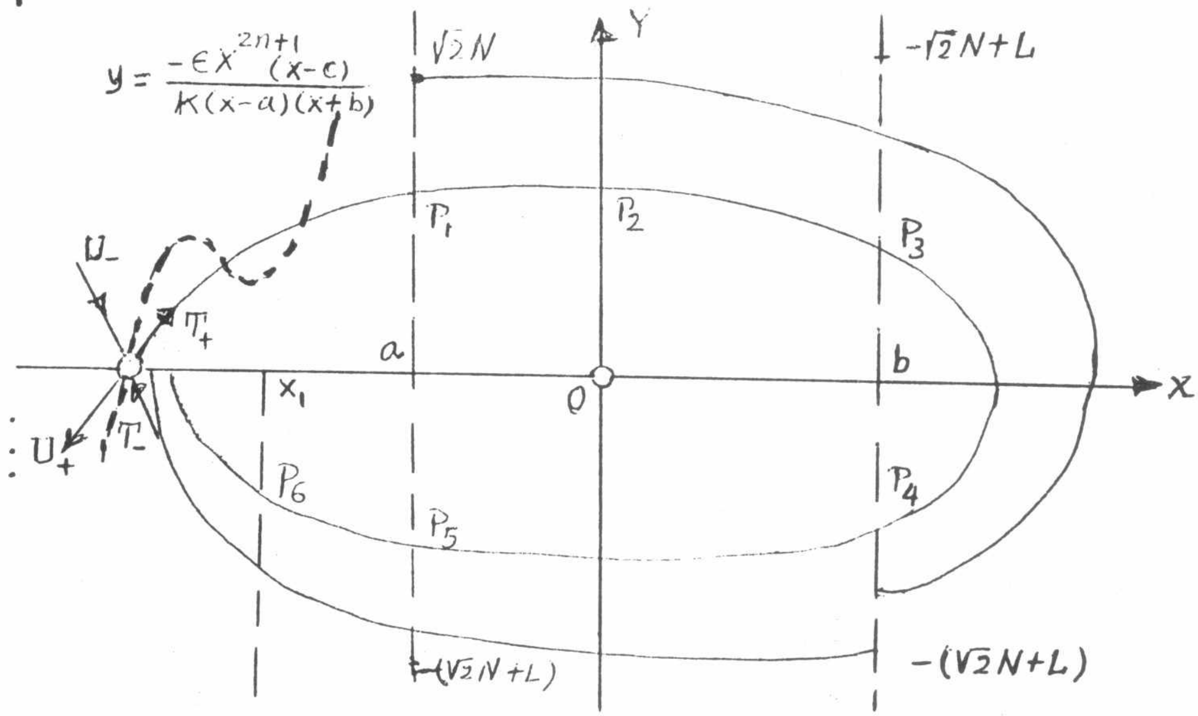

Fig.1.

Let $\lambda(x,y) = \frac{1}{2}y^2 + G(x)$,

$$\frac{d\lambda}{dt} = \frac{\partial\lambda}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial\lambda}{\partial x}\cdot\frac{dx}{dt} = -y^2 f(x) =$$

$$= -Ky^2(x-a)(x-b) .$$

$\frac{d\lambda}{dt} < 0$ when $x > b$ or $x < a$

$\frac{d\lambda}{dt} > 0$ when $a < x < b$.

Then, $\lambda(x(t),y(t))$ is decreasing for $x < a$, hence

$\lambda(P_1) < \lambda(A_o)$,i.e.,

$$\frac{1}{2}y_1^2 + G(a) < G(c),$$

$$\frac{1}{2}y_1^2 < [G(c) - G(a)] = \epsilon\int_c^a x^{2n+1}(x-c)\,dx,$$

$$y_1 < \sqrt{2}\left[\epsilon \int_c^a x^{2n+1} (x-c)\, dx\right]^{\frac{1}{2}}.$$

then $y_1 < \sqrt{2}\ N$.

Now,we have two cases,either $y_1 < N$ or $y_1 \geq N$.suppose first that $y_1 \geq N$, then by eliminating t between the two equations of the system (3) we get:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-yf(x)-g(x)}{y}$$

$$= -k\ (x-a)(x-b)- \frac{\epsilon\ x^{2n+1}(x-c)}{y} \qquad (6)$$

Then , $\frac{dy}{dx} > 0$ for $y > 0$, $a \leq x \leq 0$. Then,the trajectory $y = \overset{*}{y}(x)$ of (6) satisfying $\overset{*}{y}(a) = N$ is increasing in x for $a \leq x \leq 0$.

$$\text{Let } \phi(x) = \begin{cases} -k(x-a)(x-b) - \dfrac{\epsilon\ x^{2n+1}(x-c)}{N}, & \text{for } a \leq x \leq 0. \\ -k(x-a)(x-b), & \text{for } 0 < x \leq b. \end{cases}$$

Then,we get :

$$\phi(x) \geq -k(x-a)(x-b)- \frac{\epsilon\ x^{2n+1}(x-c)}{y}$$

for $a \leq x \leq b$ and $y \geq N$.

We denote by y(x) the solution of

$$\frac{dy}{dx} = \phi\ (x), \qquad (7)$$

satisfying the initial condition $y(a) = \sqrt{2}\ N$.

Since $y(a) = \sqrt{2}\ N > N = \overset{*}{y}(a)$, and

$$\frac{dy}{dx} = \phi\ (x) \geq \frac{d\overset{*}{y}}{dx} \quad \text{for } a \leq x \leq b\ .$$

Integrating (7) from a to b, we get :

$$y(b)-y(a) = \int_a^b \phi(x)\,dx$$

$$= -k\int_a^b (x-a)(x-b)\,dx - \frac{\epsilon}{N}\int_a^c x^{2n+1}(x-c)\,dx.$$

Hence ,
$$y(b) = \sqrt{2}\,N + K\frac{(b-a)^3}{6} + \frac{\epsilon}{N}\int_0^a x^{2n+1}(x-c)\,dx$$

$$\leq \sqrt{2}\,N + k\,\frac{(b-a)^3}{6} + \frac{G}{N},$$

hence ,by (5) ,$y(b) - \sqrt{2}\,N \leq L$, and that

$y_3 \leq y(b) \leq \sqrt{2}N + L.$

But , $G(x) \to \infty$ as $x \to \infty$, then using (iv) we have the equation

$\lambda(x,0) = \sqrt{2}N + L$ in x, has a positive root which is denoted

by ξ_0.

Since $\lambda(x,y)$ is decreasing for $x \geq b$, the trajectory which leaves P_3 must meet the x- axis between x= b and $x= \xi_0$.

Let $P_4 = (b,y_4)$ be the intersection with the half line $\{x=b, y<0\}$.

Now let

$$\Delta(\lambda) = \lambda(P_4) - \lambda(P_3) = \frac{1}{2}(y_4^2 - y_3^2)$$

since $\frac{d\lambda}{dt} = -ky^2(x-a)(x-b)$, hence ,

$$\Delta(\lambda) = -\int_{P_3}^{P_4} ky^2(x-a)(x-b)\,dt \leq 0.$$

Then ,$y_4^2 - y_3^2 \leq 0$ gives us $|y_4| \leq |y_3|$,thus we have:

$$|y_4| \leq |y_3| \leq \sqrt{2}\,N + L \tag{8}$$

For the case $y_1 < N$, we consider the trajectory $\tilde{T}(\alpha(t), \beta(t))$ of the system (3) leaving the point (a,N) , 6

also denote by t_3 the value of t such that $\alpha(t_3)$ =b, then using the previous argument, we can show that

$$|y_4| \leq |y_3| \leq \sqrt{2}N + L.$$

In addition to the hypothesis that T_+ meets the line x= b, we make an assumption that T_+ meets also the half line $\{x = x_1, y < 0\}$.

Let $P_5 = (a, y_5)$ and $P_6 = (x_1, y_6)$ be the first points of intersection of T_+ and the half lines $\{x=a, y < 0\}$ and $\{x \quad x_1, y < 0 \mid c < x_1 < a\}$ respectively. Proceeding from the point P_4 to P_5 in a similar way as from P_1 to P_3 and by using the inequality (8) , we get

$$|y_5| < \sqrt{2} \quad N + 2L \quad . \qquad (9)$$

we have

$$\frac{d\lambda}{dx} = \frac{d\lambda}{dt} \Big/ \frac{dx}{dt}, \quad \frac{d\lambda}{dt} = \frac{\partial\lambda}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial\lambda}{\partial y} \cdot \frac{dy}{dt},$$

hence , $\frac{d\lambda}{dx} = -\,y(x-a)\ (x-b)\ k.$

Integrating from P_5 to P_6 we get :

$$\lambda(P_6) - \lambda(P_5) = -\int_a^{x_1} ky(x-a)(x-b)dx$$

$$= -\int_{x_1}^{a} k\ |y|\,(x-a)(x-b)\ dx,$$

Since $y < 0$ in the interval $[x_1, a]$ and hence $|y| = -y$.

We have two cases either $|y| \geq \frac{\sqrt{2}N}{\omega}$ for every point on $\widehat{P_5P_6}$ or not.

If $|y| \geq \frac{\sqrt{2}N}{\omega}$ along $\widehat{P_5P_6}$, by using condition (ii) we get

$$\lambda(P_6) - \lambda(P_5) \leq -\frac{\sqrt{2}NK}{\omega} \int_{x_1}^{a} (x-a)\ (x-b)\ dx$$

$$= -\frac{\sqrt{2}N}{\omega}\ B\ (x_1, a) \leq -2\sqrt{2}L\ (N + \frac{L}{\sqrt{2}})$$

and $\lambda(P_6) \leq \lambda(P_5) - 2L(\sqrt{2}N + L)$.

Using $\lambda(p_5) = \frac{1}{2} y_5^2 + G(a)$ and the inequality (9) , we get

$$\lambda(p_6) = \frac{1}{2}(\sqrt{2}N + 2L)^2 + G(a) - 2L(\sqrt{2}N + L)$$

$$= N^2 + G(a)$$

$$= \epsilon\int_a^c x^{2n+1}(x - c)\,dx + \epsilon\int_0^a x^{2n+1}(x-c)\,dx$$

$$= \epsilon\int_0^c x^{2n+1}(x - c)\,dx = G(c).$$

Then,

$$\lambda(p_6) \leqslant G(c) \qquad (10)$$

If there exists at least one point of $\widehat{p_5p_6}$ such that $|y| \leqslant \frac{\sqrt{2}N}{\omega}$ at that point, then using the fact that $\dot{y}(t) > 0$ for $c < x < a$ and $y < 0$ on $\widehat{p_5p_6}$ it follows that

$$y_6 \leqslant \frac{\sqrt{2}N}{\omega}$$

Using condition (iii) , we get:

$$\lambda(p_6) = \frac{1}{2} y_6^2 + G(x_1) \leqslant \frac{N^2}{\omega^2} + G(x_1) \leqslant \epsilon\int_{x_1}^{c} x^{2n+1}(x-c)\,dx + G(x_1) = G(c),$$

which means that the inequality (10) holds.

Continuing with T_+, as λ decreases in the half plane $x \leq a$, it follows that T_+ must intersect the negative x axis at some point $(x^*, 0)$ where,

$$G(x^*) = \lambda(x^*, 0) < \lambda(p_6).$$

Let x_2 be a point such that $G(x_2) = \lambda(p_6)$ then $G(x_2) > G(x^*)$.

But $G(x)$ is decreasing in $c < x < 0$ then $c < x_2 < x^* < x_1$.

Now, we have proved that T_+ is a contracting spiral.

If T_+ does not intersect the half line $\{x = x_1, y < 0\}$, then, by constructing a function similar to $\phi(x)$, we can prove that $y \geq -(\sqrt{2N} + 2L)$ and it follows that T_+ intersects the x-axis at x^*, $x_1 < x^* < 0$ and again T_+ is a contracting spiral.

In the same way, we can prove that, if T_+ does not meet the line $x=b$, it must intersect the negative x-axis at some point x^*, $c < x^* < 0$. Again T_+ is a contracting spiral.

In addition to T_+ being a contracting spiral, the origin is an unstables critical point. Hence, there exists an annulus surrounding the origin which satisfies the hypothesis of Poincare-Bendixon Theorem (See Ref. 4)which proves the existence of at least one periodic solution.

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