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1 # OSCILLATIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

2 ## GENERATION BY SEVERAL RETARDED AND ADVANCED ARGUMENTS

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4 ABSTRACT

In this paper I study the oscillatory behaviour of equations of the forms (*) $y'(t)+qy(t)+\sum_{i=1}^{n} p_i y(t-\tau_i)=0$ and (**) $y'(t)-qy(t) - \sum_{i=1}^{n} p_i y(t+\tau_i)=0$, where $q \geqslant 0$, $p_i > 0$ and $\tau_i > 0$, are constants, $i=1,\dots,n$. It is proved that each of the following conditions (1) $p_i \tau_i \cdot \exp(1+q\tau_i) > 1$ for some $i, 1=1,2,\dots, n$, (2) $(\sum_{i=1}^{n} p_i)\tau \exp(1+q \quad)\tau > 1$, where $\tau = \min\{\tau_1, \tau_2, \dots, \tau_n)$, (3) $(\prod_{i=1}^{n} p_i)(\sum_{i=1}^{n}\tau_i)\exp(n+q\sum_{i=1}^{n}\tau_i) \quad 1$, or (4) $\{[\sum_{i=1}^{n} (q/n+p_i)\tau_i \quad]^{\frac{1}{2}}\}^2 > \frac{n}{e}$ implies that every solution of (*) or (**) oscillates. A generalization in the case where the coefficients $q \geqslant 0$, $p_i > 0$ $i=1,\dots,n$ are continuous functions of t is also presented.

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1. INTRODUCTION

In studying the oscillatory behaviour of equations of the forms

$$y'(t) + \sum_{i=1}^{n} p_i y(t-\tau_i) = 0 \qquad (1)$$

and

$$y'(t) - \sum_{i=1}^{n} p_i y(t + \tau_i) = 0, \qquad (2)$$

where p_i and τ_i, $i=1,2,\ldots,n$, are positive constants, Ladas and Stavroulakis [1] proved that each of the following conditions

(c_1) $p_i \tau_i > \frac{1}{e}$, for some i, $i=1,2,\ldots,n$,

(c_2) $(\sum_{i=1}^{n} p_i)\tau > \frac{1}{e}$, where $\tau = \min\{\tau_1, \ldots, \tau_n\}$,

(c_3) $(\prod_{i=1}^{n} p_i)^{1/n} (\sum_{i=1}^{n} \tau_i) > \frac{1}{e}$,

(c_4) $(\frac{1}{n}) (\sum_{i=1}^{n} (p_i \tau_i)^{\frac{1}{2}})^2 > \frac{1}{e}$,

implies that every solution of (1) or (2) oscillates.

In this paper, the work is extended to the equations of the forms:

$$y'(t)+qy(t) + \sum_{i=1}^{n} p_i y(t-\tau_i) = 0 , \qquad (3)$$

and

$$y'(t)-qy(t) - \sum_{i=1}^{n} p_i y(t+\tau_i) = 0 , \qquad (4)$$

where $q \geq 0, p_i > 0, \tau_i > 0, i=1,2,..,n$, are constants. It is clear that (1) and (2) are special cases of (3) and (4) when q=0 . Thus, it is expected that the derived conditions should depend on q, and should be reduced to conditions $(c_1)-(c_4)$ if q=0. The paper is terminated by a generalization to the case $q(t) \geq 0$, and $p_i(t) > 0$ are continuous functions for $i=1,2,..,n$, and by examples.

By an oscillatory solution it is meant a solution which has arbitrarily large zeros. It is also assumed that all solutions are defined for all $t > 0$.

The following two theorems are extentions of the corresponding theorems of Ladas [2] and Kusano [3], which were given for the case q=0,

<u>Theorem 1.1</u>. The first-order inequality

$$\{y'(t)+q\,y(t)+p\,y(t-\tau)\}\ \mathrm{sgn}\ y(t-\tau) \leq 0, \tag{5}$$

where $q \geq 0, p > 0$ and $\tau > 0$ are constants, has no nonoscillatory solution if and only if $p\,\tau\exp(1+q\,\tau) > 1$.

Proof. without loss of generality, let y(t) be a solution of (5) which is positive on $[t_o, \infty)$. Then we have

$$y'(t) \leq -q\,y(t) - p\,y(t-\tau)\ ,\ t \geq t_1 \tag{6}$$

where $t_1 = t_o + \tau$. Since $y'(t) < 0$, y(t) is decreasing and so $y(t) \leq y(t-\tau)$ for $t \geq t_1$. Put $w(t) = y(t-\tau)/y(t)$ and let $w = \liminf_{t\to\infty} w(t)$. We show that w is finite. Otherwise, let w be infinite. Then $\lim_{t\to\infty} w(t) = \infty$. Integrating (6) from $(t - \frac{1}{2}\tau)$ to t, we have

$$y(t)-y(t-\tfrac{1}{2}\tau) \leq -q\int_{t-\frac{1}{2}\tau}^{t} y(s)\,ds - p\int_{t-\frac{1}{2}\tau}^{t} y(s-\tau)\,ds,$$

$$\leq -\tfrac{1}{2}q\tau y(t) - \tfrac{1}{2}p\tau\, y(t-\tau),$$

which gives for $t > t_1 + \frac{1}{2}\tau$

$$\frac{y(t-\frac{1}{2}\tau)}{y(t)} - 1 \geq \tfrac{1}{2}q\tau + \tfrac{1}{2}p\tau\frac{y(t-\tau)}{y(t)}\ , \tag{7}$$

and

$$1-\frac{y(t)}{y(t-\frac{1}{2}\tau)} \geq \frac{1}{2}\, q\, \tau \frac{y(t)}{y(t-\frac{1}{2}\tau)} + \frac{1}{2}\, p\tau\, \frac{y(t-\tau)}{y(t-\frac{1}{2}\tau)} \qquad (8)$$

From (7) it follows that $\lim_{t\to\infty} y(t-\frac{1}{2}\tau)/y(t) = \infty$. But this is in contradiction with (8). Hence w is only finite.

Now, dividing (6) by y(t) and integrating from $t-\tau$ to t we get

$$-\log w(t) \leq -q\tau - p \int_{t-\tau}^{t} w(s)>ds,\ t \geq t_1$$

hence

$$\log w(t) \geq q\tau + p \int_{t-\tau}^{t} w(s)\, ds \geq q\tau + p\tau w \quad ,\ t > t_1,$$

Taking the lower limit as $t\to\infty$ we get

$\log w \geq q\tau + p\,\tau w.$

Let $F(w) = \log w - q\tau - p\tau\, w.$

Then it is clear that $F(w) \geq 0$ for some $w \geq 1$, and $\frac{dF}{dw} = \frac{1}{w} - p\tau = 0$, for $w_c = \frac{1}{p}$. Since $\frac{d^2F}{dw^2} = -\frac{1}{w^2} < 0$, then the maximum of F at the critical point w_c is nonnegative, that is

$$\log \frac{1}{pt} - 1 - q\tau \geq 0 \ , \quad \text{or} \quad p\tau \leq \exp(-1-q\tau) \ , \quad \text{or} \quad p\tau \exp(1+q\tau) \leq 1.$$

On the other hand, suppose that $p\tau \exp(1+q\tau) \leq 1$. Then as easily verified, $y(t) = \exp\left[-\left(\frac{1}{\tau}+q\right)t\right]$ is a solution of (5) . Thus the proof is complete.

By exactly the same way we can prove the next theorem, which I give its proof for completeness.

<u>Theorem 1.2.</u> The first - order inequality

$$\{y'(t) - qy(t) - p\, y(t+\tau)\}\ \mathrm{sgn}\ y(t+\tau) \geq 0, \qquad (9)$$

where $q \geq 0$, $p > 0$ and $\tau > 0$ are constants, has no nonoscillatory solution if and only if $p\tau \exp(1+q\tau) > 1$.

Proof. Without loss of generality, let y(t) be a solution of (9) which is positive on $[t_o,\infty)$. We then have

$$y'(t) \geq q\, y(t) + p\, y(t+\tau)\ ,\ t \geq t_o. \qquad (10)$$

Since $y'(t) > 0$, $y(t)$ is increasing and so $y(t+\tau) \geq y(t)$ for $t \geq t_o$, Put $w(t) = y(t+\tau)/y(t)$ and $w = \liminf_{t \to \infty} w(t)$. We show that w cannot be infinite. Suppose that w is infinite, so $\lim_{t \to \infty} w(t) = \infty$ .Integrating (10) from t to $t + \frac{1}{2}\tau$, we obtain

$$y(t + \tfrac{1}{2}\tau) - y(t) \geq q \int_t^{t+\frac{1}{2}\tau} y(s)\, ds + p \int_t^{t+\frac{1}{2}\tau} y(s+\tau)\, ds,$$

$$\geq \tfrac{1}{2} q \tau y(t) + \tfrac{1}{2} p \tau y(t+\tau) \ , \quad t > t_o$$

which gives, for $t > t_o$

$$\frac{y(t+\frac{1}{2}\tau)}{y(t)} - 1 \geq \tfrac{1}{2} q\tau + \tfrac{1}{2} p\tau \frac{y(t+\tau)}{y(t)} \ , \tag{11}$$

and $$1 - \frac{y(t)}{y(t+\frac{1}{2}\tau)} \geq \tfrac{1}{2} q\tau \frac{y(t)}{y(t+\frac{1}{2}\tau)} + \tfrac{1}{2} p\tau \frac{y(t+\tau)}{y(t+\frac{1}{2}\tau)} \ , \tag{12}$$

From (11) it follows that $\lim_{t \to \infty} y(t+\frac{1}{2}) / y(t) = \infty$, which is in contradiction with (12) , hence w is finite.

Now dividing (10) by $y(t)$ and integrating from t to $t+\tau$ we get

$$\log w(t) \geq q\tau + p \int_t^{t+\tau} w(s)\, ds \geq q\tau + p\tau w \ , \quad t \geq t_1.$$

Taking the lower limit as $t \to \infty$, we get

$$\log w \geq q\tau + p\tau w.$$

Now if we consider the function

$$F(w) = \log w - q\tau - p\tau w,$$

which is non-negative, as exactly we did in the previous theorem we arrive at the conclusion that

$$p\tau \exp(1+q\tau) \leq 1 \ .$$

On the other hand, suppose that $p\tau \exp(1+q) \leq 1$. Then, we can easily verify that $y(t) = \exp\left[\left(\frac{1}{\tau} + q\right) t\right]$ is a solution of (9).

Thus the proof is complete.

In what follows we shall study the case of several deviating argume- nts. All the following proofs are modifications of Ladas and Stavroulakis,

[1] proofs adapting them to the required generalization.

The preceding results gives :

Theorem 1.3. Every solution of equations (3) or (4) oscillates if one of the following conditions holds:

$$p_i \tau_i \exp(1+q\tau_i) > 1, \text{ for some } i, i=1, \ldots, n, \quad (13)$$

or

$$\left(\Sigma_{i=1}^{n} p_i\right)\tau \exp(1+q\tau) > 1, \tau = \min\{\tau_1, \ldots, \tau_n\} . \quad (14)$$

Proof. Otherwise, and without loss of generality, we assume that there exists an eventually positive solution $y(t)$ of (3). Then for every $j=1,2,\ldots,n$ we obtain from Eq.(3), and for t sufficiently large

$$y'(t)+q\, y(t) + p_j y(t-\tau_j) \leq 0,$$

and also, $y'(t) + q\, y(t) + (\Sigma_{i=1}^{n} p_i)\, y(t-\tau) < 0$. (16)

Hence from Theorem 1.1., neither (13) nor (14) can hold. Hence, each of (13) and (14) is a sufficient condition for the oscillation of all solutions of (3).

Similarly, if $y(t)$ is an eventually positive solution of (4), then for every $j=1,2,..,n$, we obtain from equation (4), and for sufficiently large t

$$y'(t)-q\, y(t)-p_j y(t+\tau_j) \geq 0, \quad (17)$$

and

$$y'(t)-q\, y(t)-\Sigma_{i=1}^{n} p_i y(t+\tau_i) \geq 0. \quad (18)$$

By the same arguments, Theorem 1.2 gives that (13) and (17) are in contradication, and that (14) and (18) are also in contradiction. The proof is complete.

2. RETARDED DIFFERENTIAL EQUATIONS

Theorem 2.1. Every solution of (3) oscillates if

$$\left(\prod_{i=1}^{n} p_i\right)\left(\Sigma_{i=1}^{n}\tau_i\right)^n \exp\left(n+q\sum_{i=1}^{n}\tau_i\right) > 1. \quad (19)$$

Proof. It suffices to show that if Eq.(3) have eventually positive solution then the negation of (19) holds. So, assume that $y(t)$ is a

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solution of (3) for which $y(t) > 0$, $t \geq t_o$, for sufficiently large t_o.

Choose a $t_1 > t_o$ such that $y(t-\tau_i) > 0$, $i=1,2,..,n$, for $t > t_1$.

From (3), $y'(t) < 0$ for $t > t_1$. Next choose a $t_2 > t_1$ such that

$y(t) < y(t-\tau_i)$, $i=1,2, \ldots,n$, for $t > t_2$.

Set $w_i(t) = \dfrac{y(t-\tau_i)}{y(t)}$, $i=1,2, \ldots,n$ for $t > t_2$, (20)

and

$$w_i = \lim_{t\to\infty} \inf w_i(t), \quad i=1,2,..,n. \tag{21}$$

Then $w_i(t) > 1$ and $w_i \geq 1$ for $i=1,2,...,n$. Dividing both sides of (3) by $y(t)$ for $t > t_2$, we obtain

$$\frac{y'(t)}{y(t)} + q + \Sigma_{i=1}^{n} p_i w_i(t) = 0 , \qquad =1,2,..,n.$$

Integrating both sides of the last equation from $t-\tau_k$ to t for $k=1,2,..,n$, we find that

$$\log y(t) - \log y(t-\tau_k) + q\,\tau_k + \Sigma_{i=1}^{n} p_i \int_{\tau_k}^{t} w_i(s)\, ds = 0 \tag{22}$$

We show that $w_i < \infty$ for $i=1,2,..,n$. Otherwise, assume that $w_{i_o} = +\infty$ for some $i_o=1,2,..,n$.

Hence $\lim_{t\to\infty} \dfrac{y(t-\tau_{i_o})}{y(t)} = +\infty$ (23)

From (3),

$$y'(t) + qy(t) + p_{i_o} y(t-\tau i_o) \leq 0, \qquad t > t_1.$$

If we proceed exactly as in the proof of Theorem 1.1 taking $\tau = \tau_{i_o}$ we arrive at the same contradiction. Hence all $w_i < \infty$ for $i=1,2,..,n$.

Now, Eq.(22) in view of (20) and (21), yield

$$\log w_k(t) \geq q\,\tau_k + \tau_k \sum_{i=1}^{n} p_i w_i, \qquad k=1,2,..,n.$$

Taking the lower limit as $t \to \infty$, we obtain

$$\log w_k \geq q\tau_k + \tau_k \sum_{i=1}^{n} p_i w_i, \qquad k=1,2,...,n, \tag{24}$$

. . .

and adding we find

$$\sum_{i=1}^{n} \log w_i \geq q\left(\sum_{i=1}^{n} \tau_i\right) + \left(\sum_{i=1}^{n} p_i w_i\right)\left(\sum_{i=1}^{n} \tau_i\right) .$$

Set

$$F(w_1, \ldots, w_n) = \sum_{i=1}^{n} \log w_i - q\left(\sum_{i=1}^{n} {}_i\right) - \left(\sum_{i=1}^{n} p_i w_i\right)\left(\sum_{i=1}^{n} \tau_i\right).$$

Clearly

$$F(w_1, \ldots, w_n) \geq 0 \text{ for some } w_1, \ldots, w_n \geq 1 .$$

Noting that

$$\frac{\partial F}{\partial w_i} = \frac{1}{w_i} - p_i\left(\sum_{1}^{n} \tau_i\right) = 0,$$

for

$$w_i = \frac{1}{p_i\left(\sum_{1}^{n} \tau_i\right)} \quad , \; i=1, \ldots, n.$$

At the critical point

$$\left(\frac{1}{p_1\left(\sum_{1}^{n} \tau_i\right)}, \ldots, \frac{1}{p_n\left(\sum_{1}^{n} \tau_i\right)} \right),$$

the function F has a maximum because the quadratic form

$$\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial w_i \partial w_j} a_i a_j$$

is equal to

$$- \sum_{i=1}^{n} \frac{a_i^2}{w_i^2} ,$$

Since $F(w_1, \ldots, w_n) \geq 0$, the maximum of F at the critical point should be nonnegative. That is

$$\sum_{i=1}^{n} \left\{-\log \left[p_i \left(\sum_{1}^{n} \tau_i\right)\right]\right\} - q\left(\sum_{1}^{n} \tau_i\right) - n \geq 0$$

i.e. $$-\log \left[\left(\prod_{i=1}^{n} p_i\right)\left(\sum_{1}^{n} \tau_i\right)^n\right] - q\left(\sum_{1}^{n} \tau_i\right) - n \geq 0$$

which contradicts (19). The proof is complete.

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Theorem 2.2. Every solution of equation (3) oscillates if

$$\{\sum_{i=1}^{n} [(\frac{q}{n} + p_i)\tau_i]^{\frac{1}{2}}\}^2 > \frac{n}{e} \quad . \qquad (25)$$

Proof. Otherwise there exists a solution y(t) of (3) such that for t_o sufficiently large

$$y(t) > 0 \ , \ t > t_o$$

Defining w_i, i=1,2,..,n as in Theorem 2.1, we arrive at the inequalities (24) . Using (24) and the fact that $\max_{w \geq 1} [\frac{\log w}{w}] = 1/e$, we find that

$$1/e \geq \frac{q\tau_j}{w_j} + \sum_{i=1}^{n} p_i \tau_j \frac{w_i}{w_j} \ ,$$

$$= \sum_{i=1}^{n} \frac{q\tau_j}{nw_j} \frac{w_i}{w_j} + \sum_{i=1}^{n} p_i \tau_j \frac{w_i}{w_j}$$

$$= \sum_{i=1}^{n} (\frac{q}{nw_i} + p_i)\tau_j \frac{w_i}{w_j} = \sum_{i=1}^{n} c_i \tau_j \frac{w_i}{w_j} \ ,$$

where

$$c_i = \frac{q}{nw_i} + p_i, \qquad i=1,2,..,n.$$

Adding these inequalities, we obtain

$$\frac{n}{e} \geq \sum_{i=1}^{n} c_i \tau_i + \sum_{i,j=1}^{n} (c_i \tau_j \frac{w_i}{w_j} + c_j \tau_i \frac{w_j}{w_i}) .$$

Using the fact that

$$c_i \tau_j \frac{w_i}{w_j} + c_j \tau_i \frac{w_j}{w_i} \geq 2\sqrt{c_i c_j \tau_i \tau_j} \ ,$$

the last inequality yields

$$\frac{n}{e} \geq \sum_{i=1}^{n} c_i \tau_i + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sqrt{c_i c_j \tau_i \tau_j} = (\sum_{1}^{n} (c_i \tau_i)^{\frac{1}{2}})^2.$$

Hence

$$\frac{n}{e} > (\sum_{i=1}^{n} [(\frac{q}{nw_i} + p_i)\tau_i]^{\frac{1}{2}})^2 \ ,$$

for all $w_i \geq 1$, i=1,2, ...,n, and therefore

$$\frac{n}{e} \geq \left(\sum_{i=1}^{n} \left[\left(\frac{q}{n} + p_i \right) \tau_i \right]^{\frac{1}{2}} \right)^2,$$

in contradiction with (27). The proof is complete.

3. ADVANCED DIFFERENTIAL EQUATIONS

Theorem 3.1. Every solution of the equation (4) oscillates if

$$\left(\prod_{i=1}^{n} p_i \right) \left(\sum_{i=1}^{n} \tau_i \right)^n \exp \left(n+q \sum_{i=1}^{n} \tau_i \right) > 1. \qquad (19)$$

Proof. Otherwise there exists a solution $y(t)$ of (4) such that for t_o sufficiently large

$$y(t) > 0 \ , \ t > t_o .$$

Then from (4) ,$y'(t) > 0$ for $t > t_o$. Hence $y(t+\tau_i) > y(t)$,$i=1,2,..,n$, for $t > t_o$.

Set $$z_i(t) = \frac{y(t+\tau_i)}{y(t)} , \ i=1,2,..,n \text{ for } t > t_o, \qquad (26)$$

and

$$\lambda_i = \liminf_{t \to \infty} z_i(t), \quad i=1,2,...,n. \qquad (27)$$

Then $z_i(t) > 1$ and $\lambda_i > 1$ for $i=1,2,...,n$. Dividing both sides of (4) by $y(t)$ for $t > t_o$, we obtain

$$\frac{y'(t)}{y(t)} - q - \sum_{i=1}^{n} p_i z_i(t) = 0, \qquad i=1,2,...,n.$$

Integrating the last equation from t to $t+\tau_k$ for $k=1,2,...,n$, we have

$$\log y(t+\tau_k) - \log y(t) = q\,\tau_k + \sum_{i=1}^{n} p_i \int_t^{t+\tau_k} z_i(s)\, ds, \quad k=1,2,...,n. \qquad (28)$$

We show that $\lambda_i \neq +\infty$ for any $i=1,2,...,n$. Otherwise, let $\lambda_i = +\infty$ for some $i_o = 1,2, ...,n$. Then,

$$\lim_{t \to \infty} \frac{y(t+\tau_{i_o})}{y(t)} = +\infty .$$

From Eq.(4) we have

$$y'(t)-qy(t)-p_{i_o}\, y(t+\tau_{i_o}) \geq 0, \quad t>t_o.$$

Integrating the last inequality from t to t $+\frac{1}{2}\tau_{i_o}$ and using the fact that y(t) is increasing and proceeding exactly as in the proof Theorem 1.2., with $\tau=\tau_{i_o}$ we arrive at the same contradiction. Therefore $w_i < +\infty$ for all i=1,2,..,n. Then(28), in view of (26) and (27) ,yields

$$\log z_k(t) \geq q\,\tau_k+\tau_k \sum_{i=1}^{n} p_i\,\lambda_i, \qquad k=1,2,..,n.$$

Taking the lower limit as $t\to\infty$, we obtain.

$$\log \lambda_k \geq q\,\tau_k+\tau_k \sum_{i=1}^{n} p_i\,\lambda_i, \qquad k=1,...,n. \tag{29}$$

Adding up, we get

$$\sum_{i=1}^{n} \log \lambda_i \geq q\,(\sum_1^n \tau_i) + (\sum_1^n p_i\lambda_i)(\sum_1^n \tau_i).$$

Set

$$F(\lambda_1,...,\lambda_n) = \sum_1^n \log \lambda_i - q\,(\sum_1^n \tau_i)-(\sum_1^n p_i\lambda_i)(\sum_1^n \tau_i).$$

Then, as in the proof of Theorem 2.1., we are led to a contradiction. The proof is complete.

Theorem 3.2. Every solution of equation (4) oscillates if

$$\frac{1}{n}\left(\sum_{i=1}^{n}\left[\left(\frac{1}{n}q+p_i\right)\tau_i\right]^{\frac{1}{2}}\right)^2 > \frac{1}{e} \tag{30}$$

Proof. Otherwise there exists a solution y(t) of (4) such that for t_o sufficiently large

$$y(t)>0, \quad t>t_o.$$

Define λ_i,i=1,2,...,n as in Theorem 3.1. Then, as we proved in that theorem, all the λ_i,i=1,...,n are finite. From inequality (29) and using the fact that $\max_{w\geq 1}[\log w/w] = \frac{1}{e}$ we get

$$\frac{1}{e} \geq \sum_{i=1}^{n} d_i\tau_j\,\frac{\lambda_i}{\lambda_j},$$

where $d_i = \frac{q}{n\lambda_i} + p_i$, $i=1,2,\ldots,n$.

Adding these inequalities and using the fact

$$d_i \tau_j \frac{\lambda_i}{\lambda_j} + d_j \tau_i \frac{\lambda_j}{\lambda_i} \leq 2 \sqrt{d_i d_j \lambda_i \lambda_j},$$

then as in Theorem 2.2., we are led to a contradiction.

The proof is complete.

4. GENERALIZATION

In this section we generalize the preceding results to differential equations with variable coefficients of the forms

$$y'(t) + q(t)\, y(t) + \sum_{i=1}^{n} p_i(t)\, y(t-\tau_i) = 0 \qquad (3')$$

and

$$y'(t) - q(t)\, y(t) - \sum_{i=1}^{n} p_i(t)\, y(t+\tau_i) = 0 \qquad (4')$$

where τ_i, $i=1,2,\ldots,n$, are positive constants, $p_i(t) > 0$ and $q_i(t) \geqslant 0$ are continuous functions.

Theorem 4.1. Consider equation (3') with the conditions

$$\liminf_{t\to\infty} \int_{t-\frac{1}{2}\tau_i}^{t} p_i(s)\, ds > 0, \qquad i=1,2,\ldots,n \qquad (30)$$

Then every solution of (3') oscillates if one of the following conditions holds.

$$\left(\liminf_{t\to\infty} \int_{t-\tau_i}^{t} p_i(s)\, ds\right) \exp\left(1 + \liminf_{t\to\infty} \int_{t-\tau_i}^{t} q(s)\, ds\right) > 1 \qquad (31)$$

$$\left(\liminf_{t\to\infty} \int_{t-\tau}^{t} \sum_{1}^{n} p_i(s)\, ds\right) \exp\left(1 + \liminf_{t\to\infty} \int_{t-\tau}^{t} q(s)\, ds\right) > 1,$$

where $\tau = \min\{\tau_1, \ldots, \tau_n\}$. (32)

$$\prod_{i=1}^{n} \left(\sum_{i=1}^{n} \liminf_{t\to\infty} \int_{t-\tau_j}^{t} p_i(s) ds\right) \exp\left(n + \sum_{i=1}^{n} \liminf_{t\to\infty} \int_{t-\tau_i}^{t} q(s)\, ds\right) > 1$$

(33)

$$\sum_{i=1}^{n}\left[\frac{1}{n}\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} q(s)ds\right) + \left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_i(s)ds\right)\right]$$

$$+\,2\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left[\frac{1}{n}\left(\liminf_{t\to\infty}\int_{t-\tau_j}^{t} q(s)ds\right)+\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_i(s)ds\right)\right]^{\frac{1}{2}}$$

$$\times\left[\frac{1}{n}\left(\liminf_{t\to\infty}\int_{t-\tau_j}^{t} q(s)ds\right)+\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_j(s)ds\right)\right]^{\frac{1}{2}} > \frac{n}{e} \quad . \qquad (34)$$

Proof. We present the proof when condition (33) is satisfied. The other cases can be treated in similar way. To this end suppose there exist a solution $y(t)$ of (3') such that for t_o sufficiently large,

$y(t) > 0,\ t > t_o$,

Dividing both sides of (3') by $y(t)$ and using (20) we obtain

$$\frac{y'(t)}{y(t)} + q(t) + \sum_{i=1}^{n} p_i(t) w_i(t) = 0 \ .$$

Define w_i, $i=1,2,..,n$, as in Theorem 2.1, and assume that all of them are finite. Integrating both sides of the above equation from $t-\tau_k$ to t for $k=1,2,...,n$, we find

$$\log w_k \geq \liminf_{t\to\infty}\int_{t-\tau_k}^{t} q(s)ds + \sum_{i=1}^{n} w_i\left(\liminf_{t\to\infty}\int_{t-\tau_k}^{t} p_i(s)ds\right), k=1,2,...,n$$

Adding the above inequalities, we have

$$\sum_{i=1}^{n}\log w_i \geq \sum_{i=1}^{n}\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} q(s)ds\right) + \sum_{i=1}^{n} w_i\left(\sum_{j=1}^{n}\liminf_{t\to\infty}\int_{t-\tau_j}^{t} p_i(s)ds\right).$$

Set

$$F(w_1,...,w_n) = \sum_{i=1}^{n}\log w_i - \sum_{i=1}^{n}\liminf_{t\to\infty}\int_{t-\tau_i}^{t} q(s)ds$$

$$-\sum_{i=1}^{n} w_i\left(\sum_{j=1}^{n}\liminf_{t\to\infty}\int_{t-\tau_j}^{t} p_i(s)\,ds\right).$$

Then $F(w_1,...,w_n) \geq 0$, On the other hand,

...

$$\max_{w_i \geq 1} F(w_1,\dots,w_n) = -\log \prod_{i=1}^{n} \left(\sum_{j=1}^{n} \liminf_{t\to\infty} \int_{t-\tau_j}^{t} p_i(s)\,ds \right) - \sum_{i=1}^{n} \left(\liminf_{t\to\infty} \int_{t-\tau_i}^{t} q(s)\,ds - n \right) \geq 0$$

Hence $$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} \liminf_{t\to\infty} \int_{t-\tau_i}^{t} p_i(s)ds \right) \leq \exp\left(-n- \sum_{i=1}^{n} \liminf_{t\to\infty} \int_{t-\tau_i}^{t} q(s)ds\right),$$

which is in controdiction with (29)

Finally we show that none of the $w_i, i=1,\dots,n$, can be infinite. Otherwise consider $w_{i_o} = +\infty$, for some $i=i_o$, $i=1,2,\dots,n$.

Hence $$\lim_{t\to\infty} \frac{y(t-\tau_{i_o})}{y(t)} = +\infty \tag{23}$$

From Eq.(3') and for $i= i_o$, we have

$$y'(t) + q(t)y(t) + p_{i_o}(t)\; y(t-\tau_{i_o}) \leq 0 .$$

Integrating both sides of this inequality from $t - \frac{1}{2}\tau_{i_o}$ to t and using the fact that y (t) is decreasing, we get

$$y(t)-y(t-\tfrac{1}{2}\tau_{i_o})+y(t)\int_{t-\frac{1}{2}\tau_{i_o}}^{t} q(s)ds+y(t-\tau_{i_o})\int_{t-\frac{1}{2}\tau_{i_o}}^{t} p_{i_o}(s)ds \leq 0$$

As in Theorem 1.1., and taking into account condition (23) , we are led to a contradiction and the proof is complete.

Theorem 4.2. Consider equation (4') with the conditions.

$$\liminf_{t\to\infty} \int_{t}^{t\frac{1}{2}\tau_i} p_i(s)\,ds > 0 \text{ , for } i=1,2,\dots,n \tag{30'}$$

Then every solution of (4') oscillates if one of the following conditions holds:

$$\left(\liminf_{t\to\infty} \int_{t}^{t+\tau_i} p_i(s)ds\right) \exp\left(1+\liminf_{t\to\infty} \int_{t}^{t+\tau_i} q(s)\,ds\right) > 1 \text{ .} \tag{31'}$$

$$\left(\liminf_{t\to\infty} \int_{t}^{t+\tau} \sum_{i=1}^{n} p_i(s)ds \right) \exp\left(1+\liminf_{t\to\infty} \int_{t}^{t+\tau} q(s)ds\right) > 1 \text{ ;} \tag{32'}$$

where $\tau = \min\{\tau_1,\dots,\tau_n\}$.

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$$\prod_{i=1}^{n} \Big(\sum_{i=1}^{n} \liminf_{t\to\infty} \int_t^{t+\tau_i} p_i(s)ds \; \exp\big(n+\sum_{i=1}^{n} \liminf_{t\to\infty} \int_t^{t+\tau_i} q(s)ds\big) > 1 . \tag{33'}$$

$$\sum_{i=1}^{n} \Big[\frac{1}{n} \big(\liminf_{t\to\infty} \int_t^{t+\tau_i} q(s)\,ds\big) + \big(\liminf_{t\to\infty} \int_t^{t+\tau_i} p_i(s)ds\big)\Big]$$

$$+ 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \Big[\frac{1}{n} \big(\liminf_{t\to\infty} \int_t^{t+\tau_j} q(s)ds\big) + \big(\liminf_{t\to\infty} \int_t^{t+\tau_j} p_i(s)ds\big)\Big]^{\frac{1}{2}}$$

$$\times \Big[\frac{1}{n} \big(\liminf_{t\to\infty} \int_t^{t+\tau_j} q(s)\,ds + \big(\liminf_{t\to\infty} \int_t^{t+\tau_i} p_j(s)ds\big)\Big]^{\frac{1}{2}} > \frac{n}{e} . \tag{34'}$$

Proof. We give the proof of the condition (34). The other cases can be treated similarly. Assuming contrary, there exists a solution $y(t)$ of (4') such that for t_o sufficiently large $y(t) > 0$ for all $t > t_o$.

Dividing both sides of (4') by $y(t)$ and using (26), we get

$$\frac{y'(t)}{y(t)} - q(t) - \sum_{i=1}^{n} p_i(t)\, z_i(t) = 0 . \tag{*}$$

Define $\lambda_i, i=1,\dots,n$ as in Theorem 3.1. We show that all $\lambda_i, i=1, \dots, n$, are finite. Otherwise, assume that for $i=i_o, \lambda_{i_o} = +\infty$, hence

$$\lim_{t\to\infty} z_{i_o}(t) = +\infty, \text{ i.e.}$$

$$\lim_{t\to\infty} \frac{y(t+\tau_{i_o})}{y(t)} = +\infty$$

From Eq. (4'), we have.

$$y'(t) - q(t)y(t) - p_{i_o}(t)\, y(t+\tau_{i_o}) \geq 0, \quad t > t_o$$

Integrating both sides of this inequality from t to $t+\frac{\tau_{i_o}}{2}$ and using the fact that $y(t)$ is increasing, we obtain

$$y(t+\tfrac{1}{2}\tau_{i_o}) - y(t) - y)(t) \int_t^{t+\frac{1}{2}\tau_{i_o}} q(s)ds - y(t+\tau_{i_o}) \int_t^{t+\frac{1}{2}\tau_{i_o}} p_i(s)ds \geq 0$$

As in Theorem 3.1, and taking into account condition (34') , we are led to a contradiction. Hence all $\lambda_i, i=1,\ldots,n$ are finite. Integrating both sides of equation (*) from t to $t+\tau_k$ for $k=1,2,\ldots,n$, we find after some calculation

$$\log \lambda_k \geq \liminf_{t\to\infty} \int_t^{t+\tau_k} q(s)ds + \sum_{i=1}^{n} \lambda_i \left(\liminf_{t\to\infty} \int_t^{t+\tau_k} p_i(s)ds\right), k=1,\ldots,n. \tag{35}$$

Denote by $a_j = \liminf_{t\to\infty} \int_t^{t+\tau_j} q(s)ds$, for $j=1,\ldots,n$, and

$$b_{ji} = \liminf_{t\to\infty} \int_t^{t+\tau_j} p_i(s)ds, \text{ for } 1\leq i,j\leq n,$$

(35) and the fact that $\max_{w\geq 1} [\log w/w] = \frac{1}{e}$ yield,

$$\frac{1}{e} \geq \frac{1}{\lambda_j} a_j + \sum_{i=1}^{n} b_{ji} \frac{\lambda_i}{\lambda_j} = \sum_{i=1}^{n} \frac{a_j}{n\lambda_i} \frac{\lambda_i}{\lambda_j} + \sum_{i=1}^{n} b_{ji} \frac{\lambda_i}{\lambda_j}$$

$$= \sum_{i=1}^{n} f_{ji} \frac{\lambda_i}{\lambda_j} \qquad j=1,2,\ldots,n .$$

where

$$f_{ji} = \frac{a_j}{n\lambda_i} + b_{ji}, \text{ for } 1\leq i, j\leq n.$$

Adding the last inequalities, we get

$$\frac{n}{e} \geq \sum_{i=1}^{n} f_{ii} + \sum_{\substack{i,j=1\\ i<j}}^{n} \left(f_{ji} \frac{\lambda_i}{\lambda_j} + f_{ij} \frac{\lambda_j}{\lambda_i}\right)$$

Using that fact that

$$f_{ji} \frac{\lambda_i}{\lambda_j} + f_{ij} \frac{\lambda_j}{\lambda_i} \geq 2\sqrt{f_{ji} f_{ij}} ,$$

the last inequality holds for all $\lambda_1,\ldots\lambda_n \geq 1$. Hence,

$$\frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{n}\left(\liminf_{t\to\infty} \int_t^{t+\tau_i} q(s)ds + \left(\liminf_{t\to\infty} \int_t^{t+\tau_i} p_i(s)ds\right)\right]$$

$$+ \frac{2}{n} \sum_{\substack{i,j=1\\ i<j}}^{n} \left[\frac{1}{n}\left(\liminf_{t\to\infty} \int_t^{t+\tau_j} q(s)ds\right) + \left(\liminf_{t\to\infty} \int_t^{t+\tau_j} p_i(s)ds\right)\right]^{\frac{1}{2}}$$

$$\sum \left[\frac{1}{n}\left(\liminf \int^{t+\tau_i} q(s)ds\right) + \left(\liminf \int^{t+\tau_i} p_i(s)ds\right)\right]^{\frac{1}{2}} \ldots 1$$

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But this contradicts (34'). The proof is complete.

The following examples illustrate that the conditions (13) ,(14),(19) and (25) are independent. They are chosen in such a way that only one of them is satisfied. These examples consider the equations of two time-delays only i.e. of the form :

$$y'(t) + q\, y(t)+p_1\, y(t-\tau_1) + p_2 y(t-\tau_2) = 0 \qquad (3'')$$

$$y'(t)-q\, y(t) \; -p_1\, y(t+\tau_1) - p_2\, y(t+\tau_2)= 0 \qquad (4'')$$

EXAMPLE 4.1. Take $p_1 = \frac{1}{16}$,$p_2=\frac{1}{2}$, $\tau_1 = \frac{1}{4}$, $\tau_2= 1$ and $q = \frac{1}{40}$. Then, only condition (13) is satisfied.

EXAMPLE 4.2. :

The differential equation with retarded arguments

$$y'(t)+a\, y(t)+ \exp\left[-(a+b)\,\frac{\pi}{2}\right] \; y(t- \frac{\pi}{2}) + b\, \exp\left[-(a+b)\, 2\pi\right] y(t-2\pi) = 0$$

has the oscillatory solutions

$$y_1(t) = \left[\exp-(a+b)\; t\right] \text{sint},$$

$$y_2(t) =\left[\exp-(a+b)\; t\right] \text{cost}.$$

While the differential equation with advanced arguments

$$y'(t)-ay(t)-\exp\left[(a+b)\,\frac{\pi}{2}\right] y(t+ \frac{\pi}{2})-b\, \exp\left[(a+b)\, 2\pi\right] y(t+2\pi) = 0$$

has the oscillatory solutions

$$y_1(t) =\left[\exp\, (a+b)\; t\right] \text{sint},$$

$$y_2(t) =\left[\exp\, (a+b) t\right] \cos\; t,$$

The condtion (14) now becomes

$$(p_1+p_2)\tau\, \exp(1+q\tau) =\left\{\exp\left[-(a+b)\;\frac{\pi}{2}\right] + b\, \exp\left[-(a+b)\; 2\pi\right]\right\}\left(\frac{\pi}{2}\right)\exp\left(1+a\,\frac{\pi}{2}\right)>1.$$

It is easy to see that for $0 \leqslant a<\infty$ and $0\leqslant b< \frac{2}{\pi}\left[1 + \log\frac{\pi}{2}\right]$,

the last condition is satisfied. Hence for these ranges of the parameters a and b the existance of the oscillatory solutions of each of the preceding equations is guaranteed. For $a = \frac{1}{120}$ and $b = \frac{110}{120}$ condition (14) only is satisfied.

EXAMPLES 4.3. Take $p_1 = 1$, $p_2 = \frac{1}{4}$, $\tau_1 = \frac{1}{10}$, $\tau_2 = 1$ and $q = \frac{1}{30}$. Then only condition (19) is satisfied.

EXAMPLE 4.4. Take $p_1 = \frac{1}{10e}$, $p_2 = \frac{1}{4e}$, $\tau_1 = 1$, $\tau_2 = 2$ and $q = \frac{1}{10}$. Then, only condition (25) is satisfied.

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