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Nonlinear Multiparameter Problems in Banach Space

By

M.A. Amer * B.S.H.Ahemed **

ABSTRACT

In this paper we study the problems of existence and uniqueness of solution to the nonlinear multiparameter problems of the form

$$Au + \sum_{j=1}^{n} \lambda_j B_j u = f,$$

where $A, B_j: X \to X^*$ are mappings from a reflexive Banach space X into its dual space X^*, satisfying certain monotonicity conditions, $f \in X^*$, and λ_j $(j=1,2,\ldots,n)$ are in general complex parameters.

* Department of Math., Military Technical College, Cairo, Egypt.

** Departement of Math. University Collage of Women, Ain Shams University, Cairo, Egypt.

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1. Introduction

In this paper we study the problems of existence and uniqueness of solution to the nonlinear multiparameter problems of the form

$$Au + \sum_{j=1}^{n} \lambda_j B_j u = f,$$

where A, $B_j : X \to X^*$ are mappings from a reflexive Banach space X into its dual space X^*, satisfying certain monotonicity conditions, $f \in X^*$, and λ_j $(j=1,2,\dots,n)$ are in general complex parameters.

Application of the developed theory to the nonlinear Sturm Liouville's problem for ordinary differential equations in L^p spaces is also given.
The study of multiparameter systems of equations has also been discussed as a straightforward extension to the case of a single multiparameter problem.

Our analysis in this paper is an extension, to the Banach space case, of that given by Amer [3] , Amer and Roach [4].

2. Definitions And Basic Results

Let X be a real Banach space, X^* its conjugate space. For y in X^*, x in X, denote the value of y at x by (y,x). If T is a mapping (in general nonlinear) with domaine D(T) in X and range R(T) in X^*. We recall the following definitions.

Definition 2.1. (1) The mapping $T: X \supseteq D(T) \rightarrow X^*$ is said to be monotone if

$$(Tx-Ty, x-y) \geq 0, \quad \text{for all } x,y \in D(T).$$

(2) We call that T is strictly monotone if

$$(Tx-Ty, x-y) > 0, \quad \text{for all } x,y \in D(T),\ x \neq y.$$

(3) We call that T is strongly monotone if

$$(Tx-Ty, x-y) \geq c\|x-y\|^2, \quad \text{for some } c > 0.$$

Definition 2.2. The mapping $T: X \rightarrow X^*$ with $D(T)=X$ is called coercive from X to X^* iff there exists a continuous function $\alpha: R^+ \rightarrow R$ with $\alpha(r) \rightarrow +\infty$ as $r \rightarrow +\infty$ and such that

$$(Tx, x) \geq \alpha(\|x\|)\|x\|, \quad \text{for all } x,y \in D(T).$$

Definition 2.3. The mapping $T: X \supseteq D(T) \rightarrow X^*$ is said to be hemicontinuous if T is continuous on every line segement s of D(T) (with respect to the strong topology on s and the weak topology in the range) i.e. if the function $f : R \rightarrow R$ defined by

$$f(\lambda) = (T(x+\lambda y), z), \quad x,y \in D(T), \quad z \in X, \quad \lambda \in R$$

is continuous function of λ.

For the sake of completeness we introduce without proof the following result given by Browder [1] , [2].

Theorem 2.1. Let X be a real reflexive Banach space. If $T: X \rightarrow X^*$ is monotone, hemicontinuous and coercive mapping defined on the whole of X, then the range R(T) is all of X^*.

Theorem 2.2. If in addition to the assumption of Theorem 2.1. that T is strictly monotone, then the mapping T is bijective.

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3. Main Results

We shall use Theorem 2.1. and Theorem 2.2. for establishing the existence and uniqueness of solution for two-parameter problems of the form

$$(3.1) \qquad (A + \sum_{j=1}^{2} \lambda_j B_j)x = f \in X^*,$$

where the mapping A, $B_j (j=1,2)$: $X \to X^*$ are assumed to be strongly monotone and hemicontinuous from X to X^* with domains the whole space X. Extension of the obtained results to the case of n-parameter ($n \geqslant 3$) is straightforward. Moreover, treating each equation separately we can extend our results to examine two-parameter systems of the form

$$(A_k + \sum_{s=1}^{2} \lambda_{ks} B_{ks})x_k = f_k \in X_k^*, \quad x_k \in X_k, \; k=1,2.$$

Theorem 3.1. Let X be a real reflexive Banach space and assume that the mappings A, $B_j (j=1,2)$: $X \to X^*$ satisfy the following conditions

(i) $D(\bar{A}) = D(B_j) = X, \quad j=1,2.$

(ii) $A(\theta) = B_j(\theta) = \theta, \quad j=1,2.$

(iii) A ,$B_j (j=1,2)$ are hemicontinuous mappings from X to X^*.

(iv) A, $B_j (j=1,2)$ are strongly monotone with constants c, c_j $(j=1,2)$ respectively.

Then for every $f \in X^*$ and $\lambda_1, \lambda_2 \in R$ satisfying

$$(3.2) \qquad (c + \sum_{j=1}^{2} \lambda_j c_j) > 0$$

. . .

equation (3.1) has one and only one solution $x \in X$.

<u>Proof.</u> Define a mapping $T: X \to X^*$ by

$$Tx = (A + \sum_{j=1}^{2} \lambda_j B_j)x.$$

We have by (i), $T(\theta) = \theta$.

By condition (i), (iii) it follows that

$$D(T) = D(A) \bigcap_{j=1}^{2} D(B_j) = X,$$

and T is hemicontinuous from X to X^*. Assume that $\lambda_j (j=1,2)$ are chosen such that (3.2) holds, then

$$(Tx-Ty, x-y) = (Ax-Ay, x-y) + \sum_{j=1}^{2} \lambda_j (B_j x - B_j y, x-y)$$

$$\geq (c + \sum_{j=1}^{2} \lambda_j c_j) \|x-y\|^2, \quad \text{for all } x,y \in D(T).$$

Therefore, T is strongly monotone and hence strictly monotone mapping from X to X^*. Furthermore, we have from (ii), (iv)

$$(Tx, x) = (Ax-A\theta, x-\theta) + \sum_{j=1}^{2} \lambda_j (B_j x - B_j \theta, x-\theta)$$

$$\geq (c + \sum_{j=1}^{2} \lambda_j c_j) \|x\|^2.$$

If we define a function $\alpha: R^+ \to R^+$ by

$$\alpha(\|x\|) = (c + \sum_{j=1}^{2} \lambda_j c_j) \|x\|,$$

then, T is coercive from X to X^*. Therefore by Theorem 2.1. and 2.2. applied to the mapping T we have the required result.

<u>Remark 3.1.</u> (i) Theorem 3.1. still valid if the constants

c, $c_j (j=1,2)$ are only assumed to be real numbers.

(2) If the strong monotonicity of A is replaced by only monotonicity assumption, then Theorem 3.1. still holds provided that (3.2) is replaced by the condition

$$\sum_{j=1}^{2} \lambda_j c_j > 0. \tag{3.3}$$

(3) If all the mappings A, $B_j (j=1,2)$ are supposed to be strictly monotone then, Theorem 3.1. still valid provided that $\lambda_1, \lambda_2 > 0$.

(4) If all the mappings A, $B_j (j=1,2)$ are assumed to be monotone, then we still have existence of solution of equation (3.1) provided $\lambda_1, \lambda_2 \geqslant 0$ while uniqueness of solution is not guaranteed, since in this case the mapping T is no longer strictly monotone.

(5) Assumption (ii) of Theorem 3.1. can be dropped without affecting the result of the theorem, by defining the mappings

$$A, B_j (j=1,2) : X \to X^* \text{ by}$$

$$\tilde{A}x = Ax - A\theta,$$

$$\tilde{B}_j x = B_j x - B_j \theta, \quad (j=1,2).$$

4. Practical Example

Consider the two-parameter Strum Liouville's problem for ordinary differntial equation of the form

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(4.1) $\quad (-|y'(x)|^{p-2}y'(x))' + \sum_{j=1}^{2} \lambda_j \, a_j(x)|y(x)|^{p-2}y(x)=f, \quad x\in[a,b]$

Subject to the homogeneous boundary condition

(4.2) $\quad y(b) = y(a) = 0,$

where $p\geq 2$, $\lambda_j \in R$, a_j, $f\in C[a,b]$, $j=1,2$.

Let

$$(v,u) = \int_a^b u(x)\, v(x)\, dx$$

be the natural pairing between u in $X=L^p[a,b]$ and v in $X^*=L^q[a,b]$ with $q=p/(p-1)$ and define the mappings

$A,\ B_j(j=1,2) : X\rightarrow X^*$ asfollows

$$Au = (-|u'(x)|^{p-2}u'(x))',$$

$$B_ju = a_j(x)|u(x)|^{p-2}u(x), \quad (j=1,2).$$

For each $u,\ v\in X$ we have

$$(Au-Av,u-v)= \int_a^b -(|u'(x)|^{p-2}u'(x)-|v'(x)|^{p-2}v'(x))'\,(u(x)-v(x))\,dx$$

Thus integ rating by parts we get

$$(Au-Av,u-v)= \int_a^b (|u'(x)|^{p-2}u'(x)-|v'(x)|^{p-2}v'(x))\,(u'(x)-v'(x))\,dx$$

$$= \int_a^b (|u'(x)|^{p-1}\operatorname{sgn} u'(x)-|v'(x)|^{p-1}\operatorname{sgn} v'(x))\,(u'(x)-v'(x))\,dx$$

Using the inequality

$$(x-y)\,(|x+u|^{r-1}\operatorname{sgn}(x+u)-|y+u|^{r-1}\operatorname{sgn}(y+u)) \geq |x-y|^r/2^r \quad [5].$$

we get

$$(Au-Av,u-v) \geqslant \int_a^b \| u'(x)-v'(x)\|^p / 2^{p-1} \, dx$$

$$= \| u'(x)-v'(x)\|^p / 2^{p-1} > 0$$

Therefore, A is strictly monotone mapping from X to X^*.

Similarly, we can prove that $B_j (j=1,2)$ is a strictly monotone mapping from X to X^* when assuming that

$$c_j = \min a_j(x) > 0.$$

Hence, equation (4.1) with the boundary condition (4.2) reduces to an equation of the form

$$Au + \sum_{j=1}^{2} \lambda_j B_j u = f \in X^*,$$

which by Remark 3.1. (3) has for every element $f \in X^*$ exactly one solution $u \in X$ provided that $\lambda_1, \lambda_2 > 0$.

5. Concerning The Complex Case

Now we extend the result of Theorem 3.1. by allowing the Banach space X as well as the parameters λ_1, λ_2 to be complex. To this end we introduce the following generlized definitions.

Let X be a complex Banach space, X^* its conjugate space with pairing between y in X^* and x in X denoted by (y,x) and let $T : X \to X^*$ be the mapping with domain $D(T) \subseteq X$. We denote by Re(y,x) and Im(y,x) the real and imaginary parts of (y,x), respectively.

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Definition 5.1. (1) The mapping $T : X \supseteq D(T) \to X^*$ is said to be monotone if

$$Re(Tx-Ty, x-y) \geq 0, \quad \text{for all } x,y \in D(T).$$

(2) T is said to be strictly monotone if

$$Re(Tx-Ty, x-y) > 0, \quad \text{for all } x,y \in D(T), \quad x \neq y.$$

(3) We call that T is strongly monotone if

$$Re(Tx-Ty, x-y) \geq c\,\|x-y\|^2, \quad \text{for some } c > 0.$$

Definition 5.2. The mapping $T : X \to X^*$ with domain the whole space X is called coercive from X to X^* iff there exists a continuous function $\alpha : R^+ \to R$ with $\alpha(r) \to +\infty$ as $r \to +\infty$ and such that

$$Re(Tx, x) \geq \alpha(\|x\|).\|x\|, \quad \text{for all } x \in X.$$

We are now able to state the following theorem

Theorem 5.1. Let X be a complex reflexive Banach space. Let $T : X \to X^*$ be a monotone, coercive and hemicontinuous mapping from X to X^*, then the range of T is the whole space X^*. If in addition T is strictly monotone, then T is one-to-one.

Theorem 3.1. can also be extended to the following theorem

Theorem 5.2. Let X be a complex reflexive Banach space, and let A, $B_j (j=1,2) : X \to X^*$ be mappings such that

(i) $D(A) = D(B_j) = X, \quad (j=1,2)$.

(ii) $A(\theta) = B_j(\theta) = \theta, \quad (j=1,2)$.

(iii) A is hemicontinuous from X to X^*.

(iv) A, $B_j(j=1,2)$ are strongly monotone from X to X^* with constants c, $c_j(j=1,2)$ respectively.

(v) $B_j (j=1,2)$ satisfies a Lipschitz condition from X to X^* with a Lipschitz constants $L_j (j=1,2)$.

Then for every given element $f \in X^*$ the equation

$$(5.1) \qquad Ax + \sum_{j=1}^{2} \lambda_j B_j x = f \in X^*, \; x \in X, \; \lambda_1, \lambda_2 \in C$$

has a unique solution $x_o \in X$ provided that $\lambda_j = \xi_j + i \eta_j \in C$, $j=1,2$. are such that

$$(5.2) \qquad c + \sum_{j=1}^{2} (c_j \xi_j - L_j \eta_j) > 0, \; \xi_j > 0, \quad j=1,2.$$

Remark 5.1. (i) If $\lambda_j (j=1,2) \in R$, then the requirement that the mapping $B_j (j=1,2)$ should satisfy Lipschitz condition can be relaxed to requiring that $B_j (j=1,2)$ should be hemicontinuous from X to X^*. The assertion of Theorem 5.2. then remains valid provided that condition (5.2) is replaced by the condition

$$c + \sum_{j=1}^{2} \lambda_j c_j > 0.$$

(ii) If the mapping $B_j (j=1,2)$ are assumed to be linear, then the requirement of Theorem 5.2. that $B_j (j=1,2)$ should satisfy a Lipschitz condition reduces to requiring that B_j $(j=1,2)$ should be bounded in X.

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