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MATRIX METHODS IN MAGNETOHYDRODYNAMICS

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ABSTRACT

The matrix form of magnetohydrodynamics (MHD) is presented. The approach follows the recent development of matrix methods in optics. Maxwell's equations, Ohm's law, the equation of continuity, the equation of motion with the $\vec{J}x\vec{B}$ body force and the energy equation were formulated in matrix form without using vector or tensor analysis. These equations were applied to describe the propagation of plane MHD waves in conducting fluids. Two transverse modes were obtained due to a coupling between viscous and magnetic diffusion and Alfven wave. The ordinary acoustic waves was found to split into fast, slow and intermediate magnetoacoustic waves.

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Introduction

Magnetohydrodynamics (MHD) involve the interaction of electrically conducting fluids and electromagnetic fields. The result is a body force on the fluids. The fluids being considered are continuum, that is, conducting liquids and dense ionized gases. MHD devices were employed in power generators [1], propulsion units [2] , magnetic confinement [3] and others.

MHD interactions were usually described using vector and tensor analysis [4]. This paper presents the description of MHD using matrices. Matrices become of great interest to physists when Heisenberg [5] introduced the matrix form of quantum mechanics. Their application to optics is more recent. The ray-transfer matrix [6] could now be used to describe not only the geometric optics of paraxial rays but also the propagation of a diffraction limited laser beams. Following the recent development of matrix methods in optics, the matrix form of MHD can be devised. The fluid-electromagnetic field interaction results in a body force which is expressed in terms of the stress tensor whose components can be represented by a matrix. In order to see how this arises, without using tensor analysis, the matrix form of MHD equations have been presented. Then these equations were used to descibe the propagation of MHD waves in a perfectly conducting fluid.

Matrix Form of MHD Equations

Maxwell's Equations

The electromagnetic theory is described by Maxwell's equations. Using the expressions for the matrix form of vector operations which are introduced in the Appendix, these equations can be written as

$$G^T D = \rho_e \qquad (1)$$

$$G^T B = 0 \qquad (2)$$

$$\Delta E = - \frac{\partial B}{\partial t} \qquad (3)$$

$$\Delta H = J + \frac{\partial D}{\partial t} \qquad (4)$$

where D is the displacment vector, B is the magnetic induction vector, E is the electric field intensity, H is the magnetic field intensity, ρ_e is the space charge density and J is the

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current density vector. The constitutive equations can be written as

$$D = \epsilon E \qquad (5)$$

$$B = \mu H \qquad (6)$$

where ϵ is the permittivity and μ is the permeability of the fluid which are scalar quantities.

Ohm's Law

$$A^T J = \sigma (A^T E + v^T \propto B) \quad \text{where } A^T v \text{ is the fluid velocity} \qquad (7)$$

and σ is the electrical conductivity of the fluid which is a scalar quantity. For high conductivity $\sigma \rightarrow \infty$, Ohm's law indicates that for finite J, $E = -v^T \propto B$ and the current is then determined from eq.(4) and not from eq.(7).

Continuity Equation

$$\frac{\partial \rho}{\partial t} + G^T \rho v = 0 \qquad (8)$$

where ρ is the density of the fluid. For steady flow,

$$G^T \rho v = 0$$

and for incompressible flow $G^T v = 0$

Equation of Motion

$$\frac{\partial A^T v}{\partial t} + v^T G(A^T v) = -A^T G p + A^T \Delta H \propto \mu H + A^T G \tau \qquad (9)$$

where p is the pressure (normal mechanical stress) which is considered a scalar for the present situation,

$A^T \Delta H \propto \mu H$ is the electromagnetic body force in a conducting fluid (electromagnetic stress). This term can be written as

$$A^T \Delta H \propto \mu H = - A^T G \frac{\mu H^2}{2} + H^T G(A^T \mu H) \qquad (10)$$

The first term on the right side of eq.(10) is the irrotational part of the body force and adds directly to pressure in eq.(9), and the second term is the rotational part which correspond to the tension along the magnetic field lines.

The term $A^T G \tau$ in eq.(9) is the mechanical shear stress due to the motion of a viscous fluid.

Energy Equation

$$\rho c_p \frac{dT}{dt} = \frac{dp}{dt} + k G^T G T + \phi + J^2/\sigma \qquad (11)$$

where T is the temperature, k is the thermal conductivity, ϕ is a dissipation due to shear stress, J^2/σ the joule heating and c_p is the molar heat capacity at constant pressure.

State Equation

$$p = \rho RT \qquad R \text{ is the gas constant} \qquad (12)$$

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Magnetic Transport Equation

Combining Eqs.(4),(6) and (7), then

$$A^T H = A^T J = \sigma (A^T E + v^T \alpha B)$$

Taking the curl of the above equation and use Eqs.(2)and (3) yields

$$\frac{\partial H}{\partial t} = D(M - L)H + \Delta (v^T \alpha B) \qquad (13)$$

where $D = 1/\sigma\mu$ is the magnetic diffusivity. Eq.(13) shows the transport of the magnetic field by diffusion and convection. For no motion v=0 the transport is entirely by diffusion, while for $\sigma \to \infty$ the transport is entirely by convection. The magnetic field lines is considered as being elastic and the flowing fluid drag them until they are in static equilibrium. This is the case when a magnetic field is induced which adds to the applied magnetic field. The induced field is caused by the distortion of the a applied field lines because of the fluid convecting them.

In expressing the above MHD equations, the displacement current is assumed to be negligible compared to J, the force $\rho_e E$ is negligible compared to $J^T \alpha B$ and the electric stress and energy proportional to $E^T D$ is negligible compared to $H^T B$. All velocities are small compared to the velocity of light.

Propagation of Waves in MHD

One of the consequences of electromagnetic field-fluid interactions is the possibility of wave propagation in the fluid. The approach is to assume that the wave cosists of small perturbations of the variables and then discuss plane wave propagation. Assuming a uniform perfectly conducting fluid with pressure p_o, density ρ_o and temperature T_o, at rest in a uniform magnetic field $A^T H_o$. The system is now slightly pertubed by introducing a small velocity pertubation $A^T v_1$ which gives rise to other perturbations $A^T H_1$, p_1, ρ_1, and T_1 in the magnetic field, pressure, density and temperature, respectively,

$$v = v_1 \qquad \rho = \rho_o + \rho_1$$
$$H = H_o + H_1 \qquad T = T_o + T_1 \qquad (14)$$
$$p = p_o + p_1$$

We perturb the governing equations and linearize them, so that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^T G = \frac{\partial}{\partial t}$$

Consider first, the case of an incompressible fluid in which the density remains constant. The linearized equations to(9)and

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(3) using eq.(7) for high conductivity, yield

$$\rho_o \frac{\partial v}{\partial t} = - G p_1 + \Delta H_1 \alpha \mu H_o \qquad (15)$$

$$\frac{\partial H_1}{\partial t} = \Delta v_1 \alpha \mu H_o \qquad (16)$$

Let $A^T\psi_1 = \Delta v_1$ which is the fluid vorticity and apply $\frac{\partial}{\partial t}$ curl to (15) and curl to (16), we obtain an equation for $A^T\psi_1$ from the resulting equations, thus

$$\frac{\partial^2 \psi_1}{\partial t^2} = \frac{\mu H_o^2}{\rho_o} (M - L)\psi_1 \qquad (17)$$

which is in the form of a wave equation. Assuming plane wave solution $A^T\psi_1 \propto e^{i(k^T r - \omega t)}$, where $A^T k$ is the wave vector and ω is the angular frequency, and defining the phase velocity of wave

$$V_p = \frac{\omega}{|A^T K|}$$

then $V_p^2 = (\mu H_o^2/\rho_o)\cos^2\theta = C^2\cos^2\theta$

where $C = \sqrt{\frac{\mu H_o^2}{\rho_o}}$ is the Alfven velocity. The waves are transverse, since from (8) $k^T v_1 = 0$. The velocity of the wave is thus

$$V_p = \pm C\cos\theta \qquad (18)$$

where θ is the angle between the applied field H_o and the direction of propagation. Fig.1 shows the polar diagram for the phase velocity.

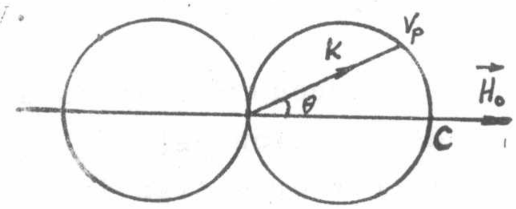

Fig.1 Polar diagram for phase velocity of propagation of incompressible waves.

Consider now the case of compressible fluid which is complicated by the full equation of continuity. Without loss of generality, the applied magnetic field is assumed to be

$$H_o = \begin{bmatrix} H_{ox} \\ H_{oy} \\ 0 \end{bmatrix}$$

and the plane wave propagates in the x-direction. The y- and z-dervatives are then zero. The linearized set of equations are

the continuity equation $\frac{\partial \rho_1}{\partial t} + (\rho_o + \rho_1)\frac{\partial V_{1x}}{\partial x} = 0 \qquad (19)$

the equation of motion

$$\begin{bmatrix} \rho_o \frac{\partial V_{1x}}{\partial t} \\ \rho_o \frac{\partial V_{1y}}{\partial t} \\ \rho_o \frac{\partial V_{1z}}{\partial t} \end{bmatrix} = \begin{bmatrix} -\frac{\partial P_1}{\partial x} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\mu H_{oy} \frac{\partial H_{1y}}{\partial x} \\ \mu H_{ox} \frac{\partial H_{1y}}{\partial x} \\ \mu H_{ox} \frac{\partial H_{1z}}{\partial x} \end{bmatrix} + \begin{bmatrix} (\xi + \frac{4}{3}\nu\rho_o)\frac{\partial^2 V_{1x}}{\partial x^2} \\ \nu\rho_o \frac{\partial^2 V_{1y}}{\partial x^2} \\ \nu\rho_o \frac{\partial^2 V_{1z}}{\partial x^2} \end{bmatrix} \qquad (20)$$

ξ : second coefficient of viscosity
ν : kinematic viscosity

the energy equation $\rho_o C_p \frac{\partial T_1}{\partial t} = \frac{\partial P_1}{\partial t} + K\frac{\partial^2 T_1}{\partial x^2} \qquad (21)$

the equatio of state $\frac{P_1}{P_o} = \frac{T_1}{T_o} + \frac{\rho_1}{\rho_o} \qquad (22)$

the magnetic transport equation

$$\begin{bmatrix} \frac{\partial H_{1x}}{\partial t} \\ \frac{\partial H_{1y}}{\partial t} \\ \frac{\partial H_{1z}}{\partial t} \end{bmatrix} = \begin{bmatrix} D\frac{\partial^2 H_{1x}}{\partial x^2} \\ D\frac{\partial^2 H_{1y}}{\partial x^2} \\ D\frac{\partial^2 H_{1z}}{\partial x^2} \end{bmatrix} + \begin{bmatrix} 0 \\ H_{ox}\frac{\partial V_{1x}}{\partial x} - H_{oy}\frac{\partial V_{1x}}{\partial x} \\ H_{ox}\frac{\partial V_{1z}}{\partial x} \end{bmatrix} \qquad (23)$$

Assuming plane wave solution for each perturbation, then nine algebraic equations for the nine amplitude quantities $A^T v_1^m, A^T H_1^m, p_1^m, \rho_1^m$ and T_1^m can be obtained. These equations are not linearly independent and the determinant of the coefficients of the variables must be zero. Eq. (24) shows that

$$\begin{bmatrix} (Dk^2+i\omega) & 0 & 0 & & & & & & \\ 0 & ik\mu H_{ox} & (\nu\rho_0 k^2+i\omega\rho_0) & & & 0 & & & \\ 0 & (Dk^2+i\omega) & ikH_{ox} & & & & & & \\ & & & -i\rho_0 k & 0 & 0 & 0 & i\omega & 0 \\ 0 & & & (\xi+\frac{4}{3}\nu\rho_0)k^2+i\omega\rho_0 & 0 & ik\mu H_{oy} & 0 & 0 & 0 \\ & 0 & & 0 & (\nu\rho_0 k^2+i\omega\rho_0) & ik\mu H_{oy} & 0 & 0 & 0 \\ & & & -ikH_{oy} & ikH_{ox} & (Dk^2+i\omega) & 0 & 0 & \\ & & & 0 & 0 & 0 & \frac{1}{P_0} & -\frac{1}{T_0} & -\frac{1}{\rho_0} \\ & & & 0 & 0 & 0 & -i\omega & 0 & (Kk^2+\rho_0 c_p i\omega) \end{bmatrix} \begin{bmatrix} H_{1x}^m \\ H_{1z}^m \\ V_{1z}^m \\ V_{1x}^m \\ V_{1y}^m \\ H_{1y}^m \\ P_1^m \\ \rho_1^m \\ T_1^m \end{bmatrix} = 0 \qquad (24)$$

there are three uncoupled terms. The first term indicates $H_{1x}=0$. The second term is a 2x2 determinant which represents two transverse modes as it is quadratic in k^2.

$$\nu D k^4 + [C_x^2 + i(\nu+D)\omega]k^2 - \omega^2 = 0 \qquad C_x = \sqrt{\frac{\mu H_{ox}^2}{\rho_0}} \qquad (25)$$

These modes are due to a coupling between viscous and magnetic diffusion and the Alfven wave. For $H=0$ ($C_x=0$) then there is no coupling,

$$(Dk^2+i\omega)(\nu k^2+i\omega)=0$$

which (k^2 is imaginary) represents pure viscous diffusion and pure magnetic diffusion. If there is no dissipation $\nu=D=0$, then k is real and the phase velocity $V_p = \pm C_x$ (+ or - sign indicates a foreward or backward wave).

The third term is a 6x6 determinant which represents a set of coupled longitudinal and transverse waves

$$\left[K\left\{\frac{1}{\rho_0} + i\frac{\omega}{\rho_0 P_0}\left(\frac{4}{3}\nu\rho_0+\xi\right)\right\}k^4 - \left\{\frac{\omega^2 K}{P_0} - i\omega c_p + \frac{\omega^2 c_p}{c^2\rho_0}\left(\frac{4}{3}\nu\rho_0+\xi\right)\right\}k^2 - i\frac{\omega^3 c_p}{c^2}\right]\left[\nu D k^4 + \left\{C_x^2 + i(\nu+D)\omega\right\}k^2 - \omega^2\right] - C_y^2(\nu k^2+i\omega)\left(\frac{\omega^2 c_p}{c^2} - \frac{i\omega K k^2}{P_0}\right)k^2 = 0 \qquad (26)$$

If there is no dissipation, that is, $\xi=\nu=D=K=0$ then

$$(C_x^2 k^2-\omega^2)\left[(C_x^2 k^2-\omega^2)(c^2 k^2-\omega^2) - C_y^2\omega^2 k^2\right] = 0 \qquad (27)$$

where c is the velocity of sound.

This equation shows that there are three waves; a transverse wave identical to Alfven wave previously obtained with phase velocity

$$V_p = \pm C_x = \pm C\cos\theta \quad \text{(intermediate wave)} \qquad (28)$$

and two longitudenal (magnetoacoustic) waves, a fast and slow ones with phase velocities

$$V_{p\,fast} = \pm\tfrac{1}{2}\left[(C^2+c^2) + \sqrt{(C^2+c^2)^2 - 4c^2C_x^2}\right] \qquad C_x = C\cos\theta \qquad (29)$$

$$V_{p\,slow} = \pm\tfrac{1}{2}\left[(C^2+c^2) - \sqrt{(C^2+c^2)^2 - 4c^2C_x^2}\right] \qquad C_x = C\cos\theta \qquad (30)$$

Fig.2 shows the polar digram for the phase velocities. For propagation perpendicular to the applied magnetic field H_o, there is only one wave, the fast magnetoacoustic wave. For propagation parallel to the magnetic field H_0, $V_{P_{fast}} = \max(c, C)$ and $V_{P_{slow}} = \min(c, C)$.

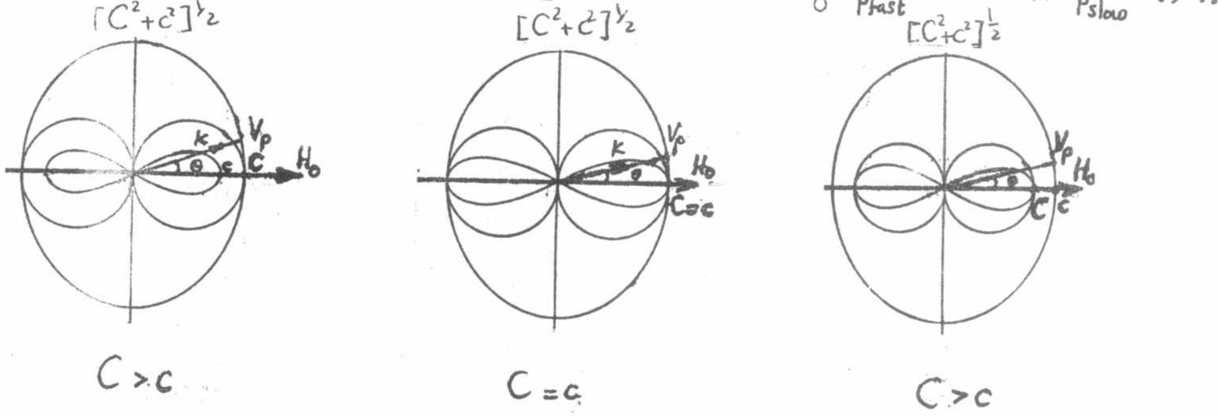

Fig.2 Polar diagram of phase velocities of non-dissipative magnetoacoustic waves.

Conculusion

Matrices provide with advantage an alternative method to problems involving Magnetohydrodynamics. Expression for the basic MHD equations were obtained without using vector or tensor analysis. Their application to the problem of propagation of MHD waves in conducting fluids gives the expressions for the phase velocities in a straight foreward way. For further study of the applications of the developed matrix methods in MHD is to consider the case of anisotrpic pressure.

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Appendix: Expression of vector operations in matrix form

A vector $\vec{H}$ of components H_x, H_y and H_z along cartesian coordinates is written as $\vec{H} = \vec{i}\,H_x + \vec{j}H_y + \vec{k}H_z$ where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the x-, y-, and z-directions.

We define two column matrices: $H = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$, $A = \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$

and their transpose: $H^T = [H_x H_y H_z]$, $A^T = [\vec{i}\ \vec{j}\ \vec{k}]$

The vector $\vec{H}$ can now be written as $\vec{H} = [\vec{i}\ \vec{j}\ \vec{k}] \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = A^T H = H^T A$

The scalar product of two vectors $\vec{H}.\vec{B}$ can be expressed as

$$\vec{H}.\vec{B} = \vec{B}.\vec{H} = H_x B_x + H_y B_y + H_z B_z = H^T B = B^T H$$

The vector product of two vectors $\vec{J}x\vec{B}$ can be expressed as

$$\begin{aligned} \vec{J}x\vec{B} &= \vec{i}(J_y B_z - J_z B_y) + \vec{j}(J_z B_x - J_x B_z) + \vec{k}(J_x B_y - J_y B_x) \\ &= J_x(-\vec{j}B_z + \vec{k}B_y) + J_y(\vec{i}B_z - \vec{k}B_x) + J_z(-\vec{i}B_y + \vec{j}B_x) \\ &= [J_x\ J_y\ J_z] \begin{bmatrix} 0 & \vec{k} & -\vec{j} \\ -\vec{k} & 0 & \vec{i} \\ \vec{j} & -\vec{i} & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = J^T \alpha B \end{aligned}$$

where $\alpha = \begin{bmatrix} 0 & \vec{k} & -\vec{j} \\ -\vec{k} & 0 & \vec{i} \\ \vec{j} & -\vec{i} & 0 \end{bmatrix}$

Similarly $\vec{J}x\vec{B} = B^T \alpha^T J$

Now, we may introduce for each matrix element a differential operator remembering that, these operators follow the distributive law. The commutative law must not be assumed.

The gradient of a scalar p is expressed as

$$grad\ p = \vec{i}\frac{\partial p}{\partial x} + \vec{j}\frac{\partial p}{\partial y} + \vec{k}\frac{\partial p}{\partial z} = [\vec{i}\ \vec{j}\ \vec{k}] \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} = [\vec{i}\ \vec{j}\ \vec{k}] \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} P = A^T G p \qquad G = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

The divergence of a vector $\vec{v}$ is expressed as

$$div\ \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \left[\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right] \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = G^T v$$

The curl of a vector $\vec{H}$ is expressed as

$$\begin{aligned} Curl\ \vec{H} &= \vec{i}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \vec{j}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \vec{k}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \\ &= [\vec{i}\ \vec{j}\ \vec{k}] \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = A^T \Delta H \end{aligned}$$

where $\Delta = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$

Now, grad div $\vec{v}$ = grad$(G^T v) = A^T G G^T v = A^T M v$

where $M = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2}{\partial z^2} \end{bmatrix}$

Curl Curl $\vec{H} = A^T \Delta(\Delta H) = A^T (M-L) H$

where $L = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ldots \end{bmatrix}$