AN ADVANCED FORMULATION FOR THE CLASSICAL METHOD OF TREFFTZ FOR 2D POTENTIAL PROBLEMS

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ABSTRACT

In this paper a variation of the usual Trefftz method is presented and applied to harmonic problems. Here, the fundamental solutions of the given problem are employed as expansion functions and their singularities are located outside the domain of the problem to avoid any handling of a singular quantities. This method has a significant computationsal advantage in that here no integrations are required. Test problems have been analysed using the proposed method and the results compared with those generated by the Regular Boundary Element Method using continuous and partially - discontinuous elements.

INTRODUCTION

The remarkable success of the Finite Element Method in design applications has led to a progrssive demand for more sophisticated design models. To date, most designs have been effected on the basis of simplified two dimensional models but there is mounting pressure, as reflected by design code trends, to use three dimensional models more extensively. However, the considerable cost increase in going from two to three dimensional analyses, incurred because of the increased number of freedom implied, stimulates interest in searching for more efficient methods than the Finite Element Method.

Since freedoms occur only on the boundary when using boundary techniques,

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such approaches are potentially more efficient than the finite element method, becuase of the likely need for fewer freedoms in a satisfactory discretization.

One boundary technique which has already been investigated in some detail is the Boundary Element Method. Although the method has been used in design applications no comprehensive formulation has yet emerged of its comparative performance in all the tackled fields over the Finite Element Method.

There are other boundary techniques available, including the classical method of Trefftz [1]. Here no integrations are needed, the solution is usually expressed as a polynomial expansion and the expansion coefficients determined by imposition of boundary conditions. In this paper a variation of this method is proposed.

An interesting feature of the Boundary Element Method is that it employs fundamental solutions of the governing equations in arriving at the boundary prescription. Being solutions of the homogeneous equations of the problem such functions are presumably efficient as expansion functions for the solution of other homageneous problems. In this paper the Trefftz method is applied to harmonic potential problems but the approximate solution is generated using fundamental solutions - with singularity located outside the domain of the problem, as in the regular boundary integral method [2]. The coefficients of the series are as usual determined by satisfaction of boundary conditions.

The method is presented for steady state, inviscid laminar fluid flow and steady state, heat conduction in two dimensions. Test problems are analysed and results compared with those given by the regular boundary element method.

THEORY

One variation of the boundary method employs a solution that satisfies the governing equations in the domain but which involves some unknown parameters determined by enforcing the boundary conditions. This technique can

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6

give good results in many practical applications, especially when the fundamental solution [3] are employed [4].

Let ϕ_o be a potential function which satisfies the governing equation (Laplace's equation) in the domain Ω of a given problem, that is

$$\nabla^2 \phi_o = 0 \qquad \text{in} \qquad \Omega \tag{1}$$

If there are "N" freedom points on the boundary, then the function ϕ_o can be approximated by a set of linearly independent functions $\phi_i(x)$ such that:

$$\phi = \sum_{i=1}^{N} \alpha_i \phi_i \tag{2}$$

where α_i are undetermined parameters.

In the advanced formulation, presented in this paper, the functions ϕ_i are taken to be the fundamental solutions that satisfy the governing equations of the problem. For two dimensional harmonic problems, they are given as:

$$\phi_i = \ln\left(\frac{1}{r_i}\right) \tag{3}$$

Where "r_i" is the distance between the source (x_i) and the field point (x).

To avoid any singularity in the solution given by equation (3), the source (x_i) is taken outside the domain of the problem (5) as in regular boundary element method [2]. Here, the singular point corresponding to a node on the boundary is relocated at an arbitrary distance and along the positive normal to the node.

Equation (2) can be written in matrix form as:

$$\phi(\underline{x}) = \sum_{i=1}^{N} \phi_i(\underline{X}_i, \underline{X})\underline{\alpha}_i = \underline{\underline{P}}(\underline{x})\underline{C} \tag{4}$$

Similarly, a system of equations for approximation of normal derivatives can be set up as :

$$\frac{\partial \phi}{\partial n}(\underline{x}) = \sum_{i=1}^{N} \frac{\partial \phi_i}{\partial n}(\underline{X}_i, \underline{X})\underline{\alpha}_i = \underline{\underline{D}}(\underline{x})\underline{C} \tag{5}$$

where

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$$\frac{\partial \phi_i}{\partial n} = - \frac{1}{r_i^2} (\hat{\underline{n}} \cdot \underline{r_i}) \qquad (6)$$

In which $\hat{\underline{n}}$ is the unit outward normal at the boundary point $\underline{x}$, $\underline{r_i}$ is the position vector from $\underline{x_i}$ to $\underline{x}$. α_i in equations (4) and (5) can be determined by satisfying the boundary conditions for the problem. This can be done by appropriately evaluating the right hand side of equation (4) or equation (5) at each node and equating to the known boundary conditions.

Let there be a node "j" on the boundary Γ where potential is prescribed as $\phi = \overline{\phi}_j$, so that from equation (4),

$$\overline{\phi}_j = \sum \phi_i \ (x_i \ , x_j) \alpha_i \qquad (7a)$$

Similarly, for another node "k" where potential derivative is given as $\frac{\partial \phi}{\partial n} = \psi_k$; Equation (5) gives :

$$\overline{\psi}_k = \sum \frac{\partial \phi_i}{\partial n} \ (x_i \ , x_k) \alpha_i \qquad (7b)$$

Equations (7a) and (7b) produce a system of "N" equations which can be written in matrix form as :

$$\underline{\phi_e} = \underline{\underline{A}} \ \underline{C} \qquad (8)$$

Now since the fundamental solutions are linearly independent at any point (x), the rows and columns of $\underline{\underline{A}}$ are also linearly independent, so that det (A) $\neq 0$ and therefore from equation (8), we get:

$$\underline{C} = \underline{\underline{A}}^{-1} \ \underline{\phi_e} \qquad (9)$$

Substituting Equation (9) in equation (4) and (5), we obtain:

$$\phi \ (\underline{x}) = \underline{P} \ (x) \ \underline{\underline{A}}^{-1} \ \underline{\phi_e} \qquad (10)$$

and

$$\frac{\partial \phi}{\partial n} \ (\underline{x}) = \underline{D} \ (x) \ \underline{\underline{A}}^{-1} \ \underline{\phi_e} \qquad (11)$$

Equations (10) and (11) are our approximate boundary solutions for potentials and their normal derives respectively. Once the boundary solution is known, solution in the interior can be determined using equation (2).

APPLICATIONS

The cases examined using the advanced Trefftz method are :
Inviscid laminar fluid flow past a circular body and a steady state heat conduction in a rectangular prism. The results are compared with these generated by the regular boundary element method using the same discretization and quadratic continuous and partially - discontinuous elements. In case of the advanced Trefftz method, the results correspond to the best location of fundamental solution singularity outside the domain of the problems [5].

Inviscid Laminer Fluid Flow Past a Circular Obstacle in a Channel

Consider the flow past an infinitely long cylinder positioned symmetrically between two flat plates of infinite dimensions. With properly specified boundary conditions it is possible to take only one quarter of the equivalent finite domain (Fig.1)

In the stream function ($\phi = \psi$) formulation of the problem, the boundary conditions are easily determined. The axis of symmetry and the upper boundary are both streamlines. The arbitrary numerical values for streamlines formed by the axis of symmetry and the cylinders, and the upper boundary are taken as $\psi = 0$ and $\psi = 2$ respectively.

The results obtained at the boundary using the advanced Trefftz and the regular boundary element methods are compared in Fig. 2. For the advanced Trefftz method, these results correspond to the best location of the singularity of the fundamental solution. Streamline $\psi = 1$ is also plotted to illustrate the interior solution.

Steady State Heat Condition in a Rectangular Prism

The domain of this problem was discretized using 16 quadratic elements with the boundary conditions as shown in Figure 3. The problem was solved using the advanced Trefftz method for the singularity location given by [6]. The temperature (ϕ = T) and the temperature normal derivative ($\frac{\partial T}{\partial n}$) are calculated at various points on the boundary and in the interior are given in table 1, where they are compared against the exact values and the values given by the regular boundary element method for the same grid.

Boundary Solution :

Function	Point	Exact Solution	Regular B.E.M.	Advanced Trefftz method
Temperature (T)	k (0.5 , 0)	35.0	35.247	34.956
	L (1.5 , 0)	25.0	25.073	25.040
	M (2.5 , 0)	15.0	14.930	14.998
	N (3.5 , 0)	5.0	4.751	5.069
Temperature normal derivative ($\frac{\partial T}{\partial n}$)	O (0 , 3.5)	10.0	10.556	10.206
	P (0 , 2.5)	10.0	9.837	9.980
	Q (0 , 1.5)	10.0	9.837	9.980
	R (0 , 0.5)	10.0	10.556	10.206

Internal Solution

Temperature (T)	X (1.0,2.0)	30.0	30.116	29.999
	Y (2.0,2.0)	20.0	20.264	20.000
	Z (3.0,2.0)	10.0	10.285	10.003

Table 1. Comparison of advanced Trefftz solution with the exact and regular boundary element solution.

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6 - 8 May 1986 , Cairo

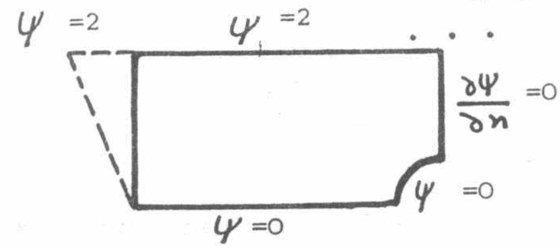

Fig.1. Boundary conditions for quarter domain (flow past a cylinder)

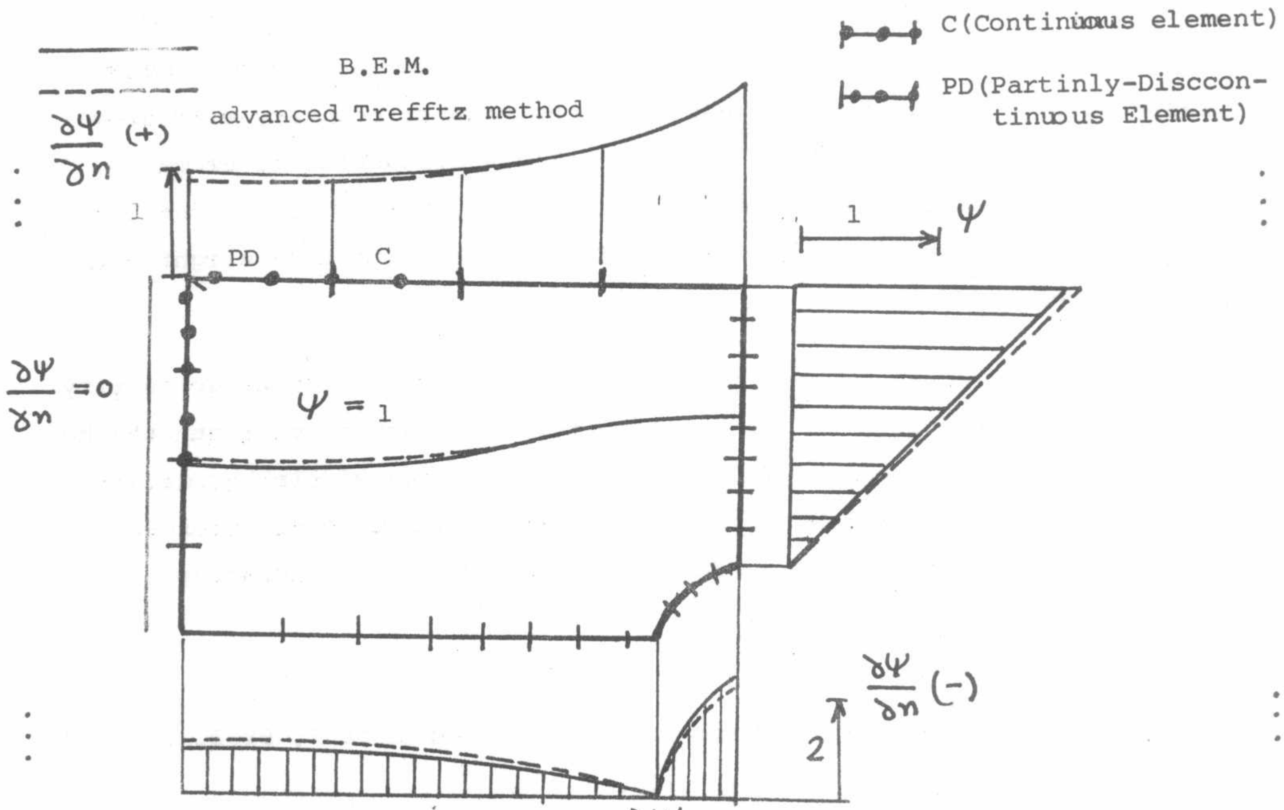

Fig.2. Computed values of ψ and $\frac{\partial \psi}{\partial n}$ using the advanced Trafftz and the regular boundary element method.

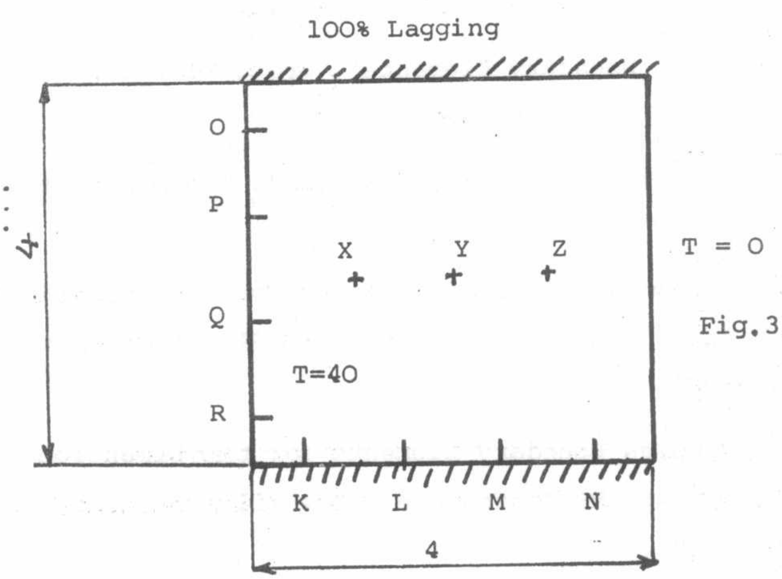

Fig.3. Heat Conduction through a prism resulting in linearly varying temperatures

DISCUSSIONS AND CONCLUSIONS

A variation of the familiar Trefftz method has been proposed for harmonic problems in which the approximate solution is sought as an expansion in terms of fundamental solutions where singular points are located outside the domain of the problem. Test problems have been analysed and results presented using the proposed method and the regular boundary element method with quadratic continuous and partially - discontinuous elements. In the cases analysed a similar discretization yielded for both solutions which were highly satisfactory in the interior but showed a slight degradation on the boundary.

In the problems analysed both methods yielded results of similar quality for a given discretization for both the regular boundary element and the advanced Trefftz methods. While both methods evidenced similar qualities of convergence it should be noted that no integrations were required in the Trefftz method implying a substantial computational advantage.

REFERENCES

1 Trefftz E. : "Ein Gregenstruck Zum Ritzchen Vergahren", Proc. 2nd. Int. Congress Appl. Mech., Zurich, 1926.

2 Patterson C. and El Sebai N.A.S. , "A Regular Boundary Method using Non-Conforming Elements for Potential Problems in three dimensions", Proc. of the 4th Int. Conference , Southampton, 112-126, 1982.

3 Brebbia C.A. : "Fundamentals of Boundary Elements", Proc. of 2nd Int. seminar of Boundary Element Methods, Southampton, 1980.

4 Brebbia C.A. : "Boundary Element Methods for Engineers", Pentech Press, 1978.

5 El Sebai, N.A.S. : "A regular Boundary Element Method For two Dimensional Stress Analysis" The Military Technical College, 1 st AME Conference, Cairo , MDB - 9.97 - 106, 1984.

6 El Sebai, N.A.S. : "Discontinuous Boundary Elements for two Dimensional Inviscid Laminar Fluid Flow, 1st AME Conference, The Military Technical College, Cairo, CA-9 . 93 - 102, 1984.