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1 # On the Oscillation of Functional-Differential Equations

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4 ## ABSTRACT

In this paper it is proved that for the RDE $y''(t)-\sum_{i=1}^{n} p_i(t)\, y(g_i(t))= f(t)$, every bounded solution is oscillatory under certain conditions imposed on the functions p_i, g_i and f for $i=1,2,..,n$.

1. INTRODUCTION

Functional-differential equations with retarded argument (RDE for short) provide a mathematical model for a physical system in which the rate of change of the system depends upon its past history. The oscillatory bechavior of RDE of order larger than or equal to 2 had been the subject of many investigations [2,4-7] just to mention a few. In this paper we consider the RDE:

$$y''(t)-\sum_{i=1}^{n} p_i(t)\, y(g_i(t)) = f(t), \qquad (1.1)$$

with the following assumptions :

(A_1) p_i, g_i and $f\varepsilon C\,[[o,\infty)\,,R]$, $f\geqslant 0$, and $p_i\geqslant o, i=1,2,..,n$, and for some index i_o, $1\leqslant i_o\leqslant n$, $p_{i_o}(t)>0$ for $t\geqslant 0$,

(A_2) $g_i(t)\leqslant t$, and $\lim_{t\to\infty} g_i(t)=\infty$ for $i=1,2,...,n$; and we shall prove that every bounded solution is oscillatory. let $\phi\varepsilon C[[0,t_o]\,,R]$ and $A\varepsilon R$ be given . Then (1.1) has a unique solution $y\varepsilon C^2[(t_o,\infty),R]$ which satisfies the initial conditions

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$$y(t)=\phi(t), \qquad 0\leqslant t\leqslant t_o \tag{1.2}$$

and

$$y'(t_o)=A \tag{1.3}$$

For more details the reader is refered to [1,2].

A solution of (1.1) is said to be oscillatory if it has arbitrary large zeros in R. Otherwise y(t) is said to be nonoscillatory. Let S denote the set of all solutions of (1.1). The following sets are introduced:

$S^{+\infty}=\{y(t)\ \varepsilon S : \lim y(t) = \lim y'(t) = +\infty$ as $t\to +\infty\}$.

$S^{-\infty}=\{y(t)\ \varepsilon S : -y(t)\ \varepsilon S^{+\infty}\}$.

$S^{o}=\{y(t)\ \varepsilon S : y(t) \neq 0$ and $\lim y(t) = \lim y'(t) = 0$ monotonically as $t\to\infty\}$.

$\tilde{S}=\{y(t)\ \varepsilon S : y(t)$ is oscillatory $\}$.

The following theorem gives a sufficient conditions for S to be the union of the four disjoint sets $S^{+\infty}$, $S^{-\infty}$, S^{o} and $\tilde{S}$.

THEOREM 1.1. If at least one of the following conditions :

(C_1) for some index k $1\leqslant k\leqslant n$, $g_k(t)$ is nondecreasing and $\int^{\infty} g_k(t)p_k(t)\,dt = \infty$,

(C_2) $\int^{\infty} g_k(t)\ f(t)\ dt =$,

holds, Then,

$$S = S^{+\infty} \cup S^{-\infty} \cup S^{o} \cup \tilde{S}.$$

Proof. Let $y(t)\varepsilon\ S-\tilde{S}$. Then $y(t)\neq 0$ for sufficiently large t, say $t\geqslant t_1$.

Case 1. $y(t)>0$ for $t\geqslant t_1$. Then, because of (A_1) and (A_2) there exists a $t_2\geqslant t_1$ such that $y''(t)>0$ for $t\geqslant t_2$. Therefore, $y'(t)$ is of fixed sign for sufficiently large t, say $t\geqslant t_3\geqslant t_2$. If $y'(t)>0$ for $t\geqslant t_3$ then $y(t)\ \varepsilon S^{+\infty}$. Indeed, $\lim_{t\to\infty} y(t) =\infty$ and $\lim_{t\to\infty} y'(t) =y'(\infty) >0$ exists. If $y'(\infty)<\infty$ then , integrating (1.1) from t_3 to t and using (C_1) or (C_2) or both, we obtain

$$y'(t) = y'(t_3)+ \int_{t_3}^{t} \sum_{i=1}^{n} p_i(x)\ y(g_i(x))\ dx + \int_{t_3}^{t} f(x)\ dx \tag{1.4}$$

$$\geqslant y(t_3) + \int_{t_3}^{t} p_k(x)\ y(g_k(x))\ dx + \int_{t_3}^{t} f(x)\ dx$$

$$\geqslant y'(t_3)+ y(g_k(t_3)) \int_{t_3}^{t} p_k(x)dx + \int_{t_3}^{t} f(x)\ dx \to \infty$$

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as $t\to\infty$. This contraduction proves that $y'(\infty)=\infty$. Hence $y(t)\in S^{+\infty}$. If, on the other hand, $y'(t)<0$ for $t\geq t_3$ then $y(t)\in S^{o}$. To prove this first observe that both $\lim_{t\to\infty} y(t)=y(\infty)$ and $\lim_{t\to\infty} y'(t)=y'(\infty)$ exist and $y(\infty)\geq 0$ while $y'(\infty)\leq 0$. We must prove that $y(\infty)=y'(\infty)=0$. Assume $y'(\infty)<0$, Then $y'(t)<y'(\infty)$, $t\geq t_3$ and therefore $y(t)\leq y(t_3)+y'(\infty)(t-t_3)\to-\infty$ as $t\to\infty$ contradicting the hypothesis that $y(t)>0$ for $t\geq t_3$. Hence $y'(\infty)=0$. Next assume that $y(\infty)>0$. Then, integrating (1.1) from t_3 to t, we get (1.4) since $y'(\infty)=0$, it follows from (1.4) that.

$$y'(t_3) = -\int_{t_3}^{\infty}\sum_{i=1}^{\infty} p_i(x)\,y(g_i(x))\,dx-\int_{t_3}^{\infty} f(x)\,dx . \tag{1.5}$$

Integrating (1.4) from t_3 to t and using (1.5) we obtain

$$y(t) = y(t_3) -(t-t_3)\Big[\int_{t_3}^{\infty}\sum_{i=1}^{n} p_i(x)\,y(g_i(x))\,dx+\int_{t_3}^{\infty} f(x)dx\Big]$$

$$+\int_{t_3}^{t}(t-x)\sum_{i=1}^{n} p_i(x)\,y(g_i(x))dx+\int_{t_3}^{t}(t-x)\,f(x)dx$$

$$= y(t_3)+\int_{t_3}^{t}(t_3-x)\sum_{i=1}^{n} p_i(x)\,y(g_i(x))dx-(t-t_3)\int_{t}^{\infty}\sum_{i=1}^{\infty} p_i(x)y(g_i(x))dx$$

$$+\int_{t_3}^{t}(t_3-x)f(x)dx-(t-t_3)\int_{t}^{\infty}f(x)dx.$$

$$\leq y(t_3)+t_3[y'(t)-y'(t_3)]-\int_{t_3}^{t} x\sum_{i=1}^{n} p_i(x)y(g_i(x))dx$$

$$-\int_{t_3}^{t} x\,f(x)\,dx.$$

$$\leq y(t_3)-t_3y'(t_3)-\int_{t_3}^{t} g_k(x)\,p_k(x)\,y(g_k(x))dx-\int_{t_3}^{t} g(x)f(x)dx \tag{1.6}$$

Choosing $t_4\geq t_3$ so large that $y(g_k(x))\geq y(\infty)/2$ for $x\geq t_4$. Then, from (1.6) we get $y(t)\leq y(t_3)-t_3y'(t_3)-y(\infty)/2\int_{t_4}^{t} g_k(x)p_k(x)dx-\int_{t_4}^{t} g_k(x)f(x)dx$

In view of (C_1) or (C_2) the right hand side of the last inequality tends to $-\infty$ as $t\to\infty$. This contradiction shows that $y(\infty)=0$. Hence $y(t)\in S^{o}$.

Case 2. $y(t)<0$ for $t\geq t_1$. A similar argumant shows that $y(t)\in S^{-\infty}\cup S^{o}$.

The proof is complete.

2. OSCILIATION OF BOUNDED SOLUTIONS

The following two theorems of [3] are also true for the general case (1.1) with a litle modification in the proofs.

THEOREM 2.1 . Assume that there exists a nonempty set of indices $K = \{k_1, k_2, \ldots, k_r\}$, $1 < k_1 < k_2 < \ldots < k_r \leq n$ such that for $t \geq t_o$

(i) $g_k \in C^1[[0,\infty), R]$, $g_k(t) < t$ and $g'_k(t) > 0$ for $k \in K$ (2.1)

(ii) $\limsup_{t \to \infty} \sum_{k \in K} \int_{g^*(t)}^{t} [g_k(t) - g_k(x)] \, p_k(x) \, dx > 1$ (2.2)

where $g^*(t) = \max_{k \in K} g_k(t)$. Then every bounded solution of (1.1) is oscillatory.

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1.1). Then, without loss of generality, $y(t) > 0$ and because of condition (A_2) there exists a $t_1 \geq t_o$ such that $y(g_i(t)) \geq 0$ for $t \geq t_1$ and $i = 1, 2, \ldots, n$. In view of (1.1) and (A_1) we have $y''(t) > 0$, $t \geq t_1$. Since $y(t) > 0$, $y''(t) > 0$ and $y(t)$ is bounded, it follows that there exists a $t_2 \geq t_1$ such that $y'(t) < 0$, $t \geq t_2$. From these observations, we conclude that $y(t)$ is concave up and decreasing for $t \geq t_2$. therefore, it lies above its tangent. That is, for any $t, x \geq t_2$,

$$y(t) + y'(t)\,(x-t) \leq y(x) \,. \tag{2.3}$$

From (2.3) and the fact that $g_k(t) \to \infty$ as $t \to \infty$ we conclude that

$$y(g_k(t)) + y'(g_k(t))\,[g_k(x) - g_k(t)] \leq y(g_k(x)) \tag{2.3a}$$

for x, t sufficiently large, say $x, t \geq t_3 \geq t_2$ and for all $k \in K$.

Multiplying (2.3a) by $p_k(x)$ and summing up for all $k \in K$, we get

$$\sum_{k \in K} p_k(x)\, y(g_k(t)) + \sum_{k \in K} y'(g_k(t))\,[g_k(x) - g_k(t)]\, p_k(x)$$

$$\leq \sum_{k \in K} p_k(x)\, y(g_k(x)) \leq \sum_{k=1}^{n} p_k(x)\, y(g_k(x)) + f(x) = y''(x) \tag{2.4}$$

Integrating (2.4), with respect to x, from $g^*(t)$ to t, for t sufficiently large, we obtain

$$\sum_{k \in K} y(g_k(t)) \int_{g^*(t)}^{t} p_k(x)\,dx + \sum_{k \in K} y'(g_k(t)) \int_{g^*(t)}^{t} [g_k(x) - g_k(t)]\, dx$$

$$\leq y'(t) - y'(g^*(t)).$$

Since $y'(t)$ increases and $g'(t) \geq 0$ this inequality, after some manipulation, becomes

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$$\sum_{k\in K} y'(g_k(t)) \int_{g^*(t)}^{t} p_k(x)dx - y'(g^*(t))\left[\sum_{k\in K}\int_{g^*(t)}^{t}\left[g_k(t)-g_k(x)\right] p_k(x)dx - 1\right]$$

$$\leqslant y'(t) \tag{2.5}$$

from (2.2) the left-hand side of (2.5) is nonnegative for sufficiently large t, while the right-hand side is negative, a contradiction. The proof is complete.

THEOREM 2.2. Assume that the hypotheses of theorems 1.1. and 2.1 are satisfied. Then $S=S^{+\infty} \cup S^{-\infty} \cup \tilde{S}$ (or equivalently $S^{o}=\phi$)

Proof. By theorem 1.1 , $S=S^{+\infty} \cup S^{-\infty} \cup S^{o} \cup \tilde{S}$. Let $S^{o}\neq\phi$ and $y(t)\in S^{o}$. Then y(t) is a bounded solution of (1.1) and by theorem 2.1 it should ascillate. This contradicts the defiention of S^{o}. Hence S^{o} is empty and the proof is complete.

COROLLARY 2.1. Consider the RDE

$$y''(t) - p(t)y(t-\tau) = f(t), \tag{2.6}$$

where $p(t) >0$ and continous, $f(t)\geqslant 0$ and continuous and $\tau>0$ constant and

$$\lim_{t\to\infty}\sup \int_{t-\tau}^{t} (t-x)\ p(x)\ dx > 1 \tag{2.7}$$

then,

$$S= S^{+\infty} \cup S^{-\infty} \cup \tilde{S} .$$

In particular, every bounded solution of (2.6) is oscillatory.

Proof . Take $n=1; g(t)=t-\tau$, $g^*(t)=t-\tau$. by Theorem (2.2) $S^{o}=\phi$ and since $S^{+\infty}\cup S^{-\infty}$ consists of unbounded solutions it follows that every bounded solution of (2.6) oscillates.

EXAMPLE 2.1. Consider the RDE.

$$y''(t) - a\ y(t-1) - by(t-2) - Cy(t-\tfrac{1}{2}) - dy(t) = t \tag{2.8}$$

where a,b,c,d are constants such that

$0<a<2$, $0 <b< \tfrac{1}{2}$, $0 <c< 1$, $d\geqslant 0$

$a+b >2$,

The hypotheses of Theorem 1.1 are satisfied with $g_k(t) = t-1$ and $p_k(t) = a$.
Hence, $S=S^{+\infty} \cup S^{-\infty} \cup S^{o} \cup \tilde{S}$.

Also the hypotheses of Theorem 2.1 are satisfied with $K=\{1,2\}$,

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$p_1(t)=a, p_2(t)=b,\ g_1(t)=t-1,\ g_2(t)=t-2,\ g^*(t)=t-1, g^{\acute{*}}(t)=1$. In fact,

$\int_{t-1}^{t} (a+b)(t-s)ds= \frac{a+b}{2} >1$ and the condition (2.2) is satisfied. Hence $S^o=\phi$ and therefore $S= S^{+\infty} \cup S^{-\infty} \cup \tilde{S}$.

EXAMPLE 2.2. Consider the RDE.

$$y''(t)-(k+1)y(t-\pi)-K\ y(t) = 0\ ,\ K \geqslant 0 \tag{2.9}$$

Then

$$\int_{t-\pi}^{t} (K+1)(t-s)ds = \frac{k+1}{2}\pi^2 > 1$$

and by Theorem 2.1 . every bounded solution of (2.9) is oscillatory. It is easily seen that Eq. (2.9) has the bounded oscillatory solutions C_1 cost $+C_2$ sint for any real numbers C_1 and C_2.

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