MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

ON PRESERVING THE STABILITY OF A PERTURBED HURWITZ POLYNOMIAL

by

Mohammad B. Argoun*

ABSTRACT

Necessary and sufficient conditions for a perturbed polynomial to remain Hurwitz are given. The conditions do not require a priori knowledge of the bounds on coefficient perturbations and allow the designer maximum freedom in allocating different weights to various coefficients to reflect different levels of uncertainty in the coefficients. The conditions are an extension of a previous result of the author in which sufficient conditions for the same problem were obtained.

1. INTRODUCTION

Given a Hurwitz polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \tag{1}$$

Let the coefficients a_i, i=0, ..., n be subject to unknown but bounded perturbations δa_i, i=0, ..., n; i.e.

$$|\delta a_i| \leq \Delta a_i, \quad i=0, \dots, n \tag{2}$$

where Δa_i are known numbers. Maximum allowable perturbations Δa_i for the perturbed polynomial

$$\tilde{P}(s) = (a_n \pm \delta a_n)\, s^n + (a_{n-1} \pm \delta a_{n-1})\, s^{n-1} + \dots + (a_0 \pm \delta a_0) \tag{3}$$

to remain Hurwitz were obtained by Kharitonov (1978). His result asserts that the perturbed polynomial will remain Hurwitz if each of four other polynomials of the same order are Hurwitz. Using a Nygist type analysis, Yeung (1983) developed sufficient conditions for the stability of the perturbed polynomial in terms of a Routh test on a polynomial of order 2n. Recently (Argoun, 1986), sufficient conditions based on the principle of the argument were obtained for the stability of a perturbed poly-

*Assistant Professor, University of Wisconsin--Milwaukee, P. O. Box 784 Milwaukee, Wisconsin 53201

nomial. The method has the advantage of giving the designer the flexibility of placing different emphasis on different coefficients so as to allow maximum error bounds on the most uncertain coefficients. In this note, while retaining the advantages of the method, we show that a slight modification will yield necessary and sufficient conditions which will give the maximum uncertainty bounds possible. The development of the conditions follows the analysis given in (Argoun 1986).

2. DEVELOPMENT

Let the polynomial P(s) be written as:

$$P(s) = R(s) + Q(s) \tag{4}$$

where

$$R(s) = a_0 + a_2 s^2 + a_4 s^4 + \dots + a_m s^m \qquad Q(s) = a_1 s + a_3 s^3 + a_5 s^5 + \dots + a_\ell s^\ell \qquad \begin{cases} m=n,\ \ell=n-1 & n \text{ even} \\ m=n-1,\ \ell=n & n \text{ odd} \end{cases} \tag{5}$$

i.e., R(s) contains the even-power terms of P(s) and Q(s) contains the odd-power ones. Let ω_k, $k=1, 2, \dots, m/2$ be the frequencies of intersection of the polynomial $R(j\omega)$ with the imaginary axis, i.e., ω_k are the solutions of

$$R(j\omega) = 0 \tag{6}$$

If perturbations are to occur in the odd coefficients only, then the maximum bounds on allowable perturbations will be determined by the following theorem

THEOREM (ARGOUN 1986)

A necessary and sufficient condition on the allowable perturbation in the odd-coefficients to retain stability of the perturbed polynomial is:

$$\max_{\substack{\Delta a_i \\ i=\text{odd}}} \ |\Delta Q(j\omega_k)| \le |Q(j\omega_k)| \ \forall\ \omega_k \tag{7}$$

where

$$\Delta Q(\omega_k) = \Delta a_1\, \omega_k + \Delta a_3\, \omega_k^3 + \dots + \Delta a_\ell\, \omega_k^\ell \tag{8}$$

If perturbations in both the odd- and even-power coefficients are allowed, we form the two odd-coefficient polynomials:

$$Q_1(s) = Q(s) - \Delta Q(s) \tag{9}$$

$$Q_2(s) = Q(s) + \Delta Q(s) \tag{10}$$

where $\Delta Q(s)$ is an odd-coefficient polynomial satisfying (8)(Figure 1.). Let the intersection frequencies of the polynomials $Q_1(j\omega)$ and $Q_2(j\omega)$ with the real axis be $\omega_{\ell_{min}}$, $\omega_{\ell_{max}}$ where the subscript ℓ indicates the number of intersection frequencies of $Q_1(s)$ and $Q_2(s)$ with the real axis which depends on the order of the polynomial. Let the odd-coefficient perturbations be chosen such that they satisfy the necessary and sufficient condition (7). Then theorem 2 in (Argoun 1986) gives a sufficient condition on the even-coefficient perturbations so that the polynomial $\tilde{P}(s)$ remains Hurwitz. In the following theorem a slight modification of the above condition gives necessary and sufficient conditions for perturbed polynomial stability.

THEOREM

The perturbed polynomial will be Hurwitz, if the complementary set of even-coefficient perturbations satisfies:

$$|\Delta R(j\tilde{\omega}_\ell)| \leq |R(j\tilde{\omega}_\ell)| \quad \forall\, \tilde{\omega}_\ell : \omega_{min_\ell} \leq \tilde{\omega}_\ell \leq \omega_{max_\ell} \tag{11}$$

PROOF

From the principle of the argument the necessary and sufficient condition for $\tilde{P}(s)$ to have the same number of unstable roots as does $P(s)$ (in this case zero) is that no crossing of origin occurs due to the perturbation. Condition (11) is obviously a sufficient condition for this to occur. On the other hand, if condition (11) is violated then perturbations Δa_i, $i=0, 2, \ldots$ and a frequence $\tilde{\omega}_\ell$ can be found in the permissible range such that

$$|\Delta R(j\tilde{\omega}_\ell)| = |R(j\tilde{\omega}_\ell)| \tag{12}$$

The sign of these perturbations can be chosen such that

$$R(j\tilde{\omega}_\ell) + \Delta R(j\tilde{\omega}_\ell) = 0 \tag{13}$$

This indicates that one real root is at the origin or two complex roots are on the imaginary axis. Further increase of the perturbations in the same direction (i.e. violating condition (11)) will ensure crossing of the origin.

3. EXAMPLE

To illustrate the advantage gained by introducing the necessary and sufficient condition, consider the following example (Yeung 1983, Argoun 1986). The nominal polynomial $P(s)$ is given by:

$$P(s) = s^6+14.0\, s^5+8025\, s^4+251.25\, s^3+502.75\, s^2+667.25\, s+433.5 \tag{14}$$

Yeung's bounds are:

on odd-coefficients: $\Delta a_5 = 1.4,\ \Delta a_3 = 15.075,\ \Delta a_1 = 33.36$ (15)

on even-coefficients: $\Delta a_6 = 0.1,\ \Delta a_4 = 5.6176,\ \Delta a_2 = 25.137,$

$\Delta a_0 = 86.7$ (16)

Using the sufficient condition in (Argoun 1986) with the bounds on odd-coefficients perturbations retained, the bounds on the

$$\Delta a_6 = 0.12,\ \Delta a_4 = 6.00,\ \Delta a_2 = 38.28,\ \Delta a_0 = 86.7 \quad (17)$$

This together with the bounds in (11) constituted a set of maximum allowable bounds according to the sufficient condition. The "worst" perturbed polynomial in this case was:

$$\tilde{P}(s) = 0.88\ s^6+15.4\ s^5+86.25\ s^4+236.175\ s^3+464.47\ s^2 + 700.61\ s+520.2 \quad (18)$$

$$\text{with roots} \quad \begin{array}{l} -1.88052 \pm j\ 0.53064 \\ -0.665 \times 10^{-2} \pm j\ 2.01074 \\ -3.89574 \\ -9.82990 \end{array} \quad (19)$$

The new necessary and sufficient condition for this example is

$$[\Delta a_6\ \omega_\ell^6+\Delta a_4\ \omega_\ell^4+\Delta a_2\ \omega_\ell^2+\Delta a_0] \leq |-\omega_\ell^6+80.25\ \omega_\ell^4-502.75\ \omega_\ell^2+433.5|$$

$$\forall\ \omega_{\ell_{min}} \leq \omega_\ell \leq \omega_{\ell_{max.}} \quad (20)$$

with ω_ℓ lying in two frequency bands, namely;

$$1.6534 \leq \omega_1 \leq 2.00515$$

$$3.3639 \leq \omega_2 < 4.2899 \quad (21)$$

To test condition (20) we form table (1) below for the proposed bounds. The bounds can then be increased until the limiting case in (20) is reached at some frequency. This provides one set of maximum allowable bounds. To illustrate, let us increase the even-coefficient bounds beyond those obtained in (15) and (17) to:

$$\Delta a_6 = 0.14,\ \Delta a_4 = 6.2,\ \Delta a_2 = 38.28,\ \Delta a_0 = 92.32 \quad (22)$$

We then form table (1) as shown below.

Low Frequency Band			High Frequency Band		
ω_1	$\lvert R(j\omega_1)\rvert$	$\lvert\Delta R(j\omega_1)\rvert$	ω_2	$\lvert R(j\omega_2)\rvert$	$\lvert\Delta R(j\omega_2)\rvert$
1.6534	361.5821	246.1618	3.3639	3571.3433	1522.2418
1.700	373.3290	258.1115	3.6000	5219.9961	1934.5402
1.800	368.9898	286.1940	3.8000	6896.0700	2359.3986
1.900	382.6474	317.8962	4.0000	8837.5000	2865.4400
2.000	357.5000	353.6000	4.2000	11047.3187	3465.2952
2.00515	355.5888	355.5546	4.2899	12127.4878	3769.1995

Table 1.

By examining table 1 it is clear that the limiting case for the necessary and sufficient condition could be reached within the low frequency band at $\omega_1 = 2.00515$. To check that these indeed are the maximum allowable bounds for stability, we calculate the roots for the "worst" perturbed polynomial, which in this case is:

$$\tilde{P}(s) = 0.86\, s^6+15.4\, s^5+86.45\, s^4+236.175\, s^3+464.47\, s^2 +700.61\, s+525.82 \quad (23)$$

with the roots $-1.86467 \pm j\, 0.58572$
$-4.0 \times 10^{-5} \pm j\, 2.00515$
-3.85728
-10.32027

We have thus increased the allowable bounds beyond those obtained from the sufficient condition by up to 16.7% for some coefficients. Two roots now lie almost exactly on the imaginary axis with their imaginary part equal to the intersection frequency ω_1. It should be noted that the new bounds given by (15) and (22) violate the sufficient condition given earlier (Argoun 1986) in the high frequency band. The minimum absolute value of the polynomial $R(j\omega_2)$, i.e., $\min|R(j\omega_2)| = 3571.3433$ is not larger than the maximum value of the perturbation polynomial, $\max.|\Delta R(j\omega_2)| = 3769.1995$ as the sufficient condition would require.

4. CONCLUSION

A simple and powerful test that constitutes necessary and sufficient conditions for the stability of a perturbed polynomial is presented in this paper. The new conditions are an extension of sufficient conditions given earlier by the author (Argoun 1986) to solve the same problem. The only comparable result to the author's knowledge is that of Kharatinov (1978) where the uncertainty bounds are assumed known. The new result does not assume a priori knowledge of the uncertainty bounds and conditions are given which have to be satisfied by all stable combinations of coefficient perturbations. Therefore different combinations of coefficient perturbations can be formed and the designer has the freedom to increase the error margin on more uncertain terms and tighten the bounds on other ones. On the other hand if the inequalities are pushed to the limit, a set of maximum allowable perturbations will be obtained. Since all sets of allowable perturbations combinations have to satisfy conditions (7) and (11), these conditions can be thought of as characterizing the structure of allowable perturbations. In order to obtain the maximum allowable perturbations from Kharitonov's theorem, Barmish (1984) and Bialas (1984) had to assume that the relative magnitudes of the perturbations in different coefficients are known a priori. The problem is then formulated in terms of one parameter which is maximized to obtain the maximum allowable perturbations before Kharitonov's conditions are violated. In our case such an a priori knowledge is not required and the only limitation is that the odd-coefficient perturbations are grouped and specified together while the even ones are subsequently grouped and maximized together. From the point of view of the freedom to manipulate the coefficient perturbations in order to accommodate different degrees of uncertainties the present result seems to present much more flexibility than is currently available.

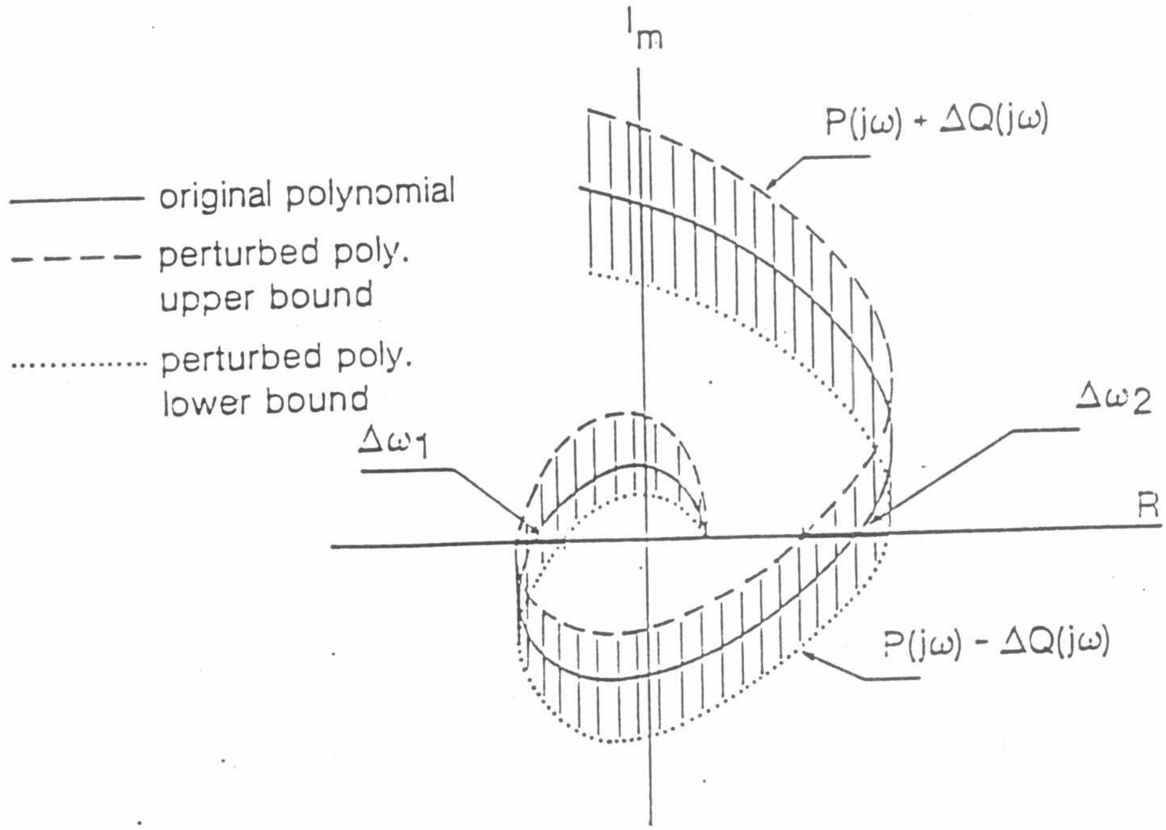

Figure 1. Polar Plot of P(s) with odd-coefficient perturbations only

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