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APPLICATION OF EXPECTATION MAXIMIZATION ALGORITHM TO THE SEQUENCE ESTIMATION OF M-ARY PHASE SHIFT KEYING SIGNALS OVER RAYLEIGH FADING CHANNEL

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Abstract

Maximum likelihood (ML) is an optimal parameter estimation technique. When applying this technique to the problem of modulation sequence estimation, it is found that the resulting estimate is unpractical due to the unavailability of an efficient way to perform the required maximization. The expectation-maximization (EM) algorithm is a statistical mean that can provide maximum likelihood parameter estimation. In this paper, this algorithm is applied to the problem of estimation of the modulation sequence of M-ary Phase Shift Keying (MPSK) signals contaminated with additive white Gaussian noise and traveled over Rayleigh fading channel. The algorithm obtains a suitable iterative mean for computing ML estimate of the sequence and performs the task that has been previously too complex to perform. Simulation experiment is performed to validate the theoretical developments and to measure the performance of the algorithm. The performance of the algorithm is measured in terms of the symbol error probability.

KEY WORDS

Wireless Communications, Fading, and Signal Processing

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I. Introduction

The EM algorithm is an attractive algorithm introduced in the statistics literatures as a general approach for iterative maximization of likelihood functions. It has applications in many estimation problems [1-3]. One of the reasons for its attractiveness is that, it provides a numerical method for obtaining maximum likelihood estimates that might not be readily available otherwise. The main purpose of this paper is to apply the EM algorithm to the problem of ML estimation for modulation sequence of fading M-ary PSK signals to obtain optimum estimate. Obtaining optimum sequence estimate involves two steps: (1) computing the likelihood function; (2) maximization over the set of all admissible sequences. When random parameters are involved, (which is our case in fading channel) evaluation of the likelihood function may require computing the expectation over the joint statistics of the vector that contains the random parameters. This task is not analytically intractable. Moreover, the likelihood function is a nonlinear function of the modulation sequence, which makes the maximization step computationally infeasible, especially for long sequences. One of the channels that face the above difficulty is the fading channel due to involving two additional random parameters besides the random sequence of the signal. These random parameters are the fading amplitude, which has a Rayleigh distribution, and the fading phase, which has a uniform distribution. Therefore, applying the EM algorithm to this problem can provide a solution that can be implemented.

The paper is organized as follows. The EM algorithm is described briefly in section II. Section III describes the mathematical formulation of the algorithm that results from application of the EM algorithm to estimate the modulation sequence. In section IV, simulation experiment is presented to demonstrate the performance of the algorithm. Finally, the conclusion is presented in section V.

II. The EM Algorithm

Let $\mathbf{r} \in R$ denotes the observed data and it is required to estimate set of parameters $\mathbf{b} \in B$. Then the ML estimate of $\mathbf{b}$, $\mathbf{b}$, can be obtained as a solution to the equation

$$\mathbf{b} = \arg \max_{\mathbf{b} \in B} p(\mathbf{r}/\mathbf{b}) \tag{1}$$

where $p(\mathbf{r}/\mathbf{b})$ is the conditional density of the observed data given the parameter vector to be estimated. In many cases obtaining a closed form solution to the optimization problem in (1) is difficult. Moreover, for some cases finding a closed form expression for this conditional density is not exist. In such situations, the EM algorithm may provide an iterative solution to the ML problem. The EM algorithm based solution proceeds as follows.

Assume that, instead of the data $\mathbf{r}$, a data $\mathbf{x} \in X$ can be accessed and it is related to $\mathbf{r}$ by noninvertible, many to one transformation, and its conditional density $p(\mathbf{x}/\mathbf{b})$ can be obtained. In the statistics literature, the two sets of data $\mathbf{r}$ and $\mathbf{x}$ are known as the *incomplete* data and the *complete data*, respectively. The EM algorithm uses the log-likelihood function of the complete data in iterative two-step procedure, which converges to the ML estimate (under some conditions) [4], [5]. These two steps are called the Expectation step (E-step) and the Maximization step (M-step). In the expectation step, an evaluation of the conditional expectation of the function $\log p(\mathbf{x}/\mathbf{b})$, given the data **r** and the most recent estimate $\mathbf{b}^i$ of **b** is required. In the maximization step, the conditional expectation computed in the E-step is maximized with respect to the parameter vector **b**. Maximizing the value of **b** is the new estimate, $\mathbf{b}^{i+1}$, of **b**. This estimate is then used in the E-step to produce the new conditional expectation and the procedure repeats until the algorithm converges. The latest estimate of **b** when the algorithm converges is considered the final estimate. Mathematically, the two step procedure at the *i*th iteration is written as

(1) E-step: Compute $U(\mathbf{b}/\mathbf{b}^i) = E[\log p(\mathbf{x}/\mathbf{b}) / \mathbf{r}, \mathbf{b}^i]$, and

(2) M-step: Solve $\mathbf{b}^{i+1} = \arg\max_{\mathbf{b}} U(\mathbf{b}/\mathbf{b}^i)$

where $\mathbf{b}^i$ is the parameter vector estimate at the *i*th iteration.

III. Mathematical Formulation

Consider the transmission of MPSK signals over Rayleigh fading channel. The transmitted signal in the interval $kT_s \le t \le (k+1)T_s$ has the form [6]

$$s_k = \sqrt{2E_s}\, e^{j\phi_k} \tag{2}$$

where ϕ_k is the transmitted phase of the *k*-th symbol which takes one of *M* uniformly distributed values $\phi_k \in \frac{2\pi(k-1)}{M}$; $k = 1,2,...,M$, E_s is the signal power, and T_s is the symbol duration of the MPSK signal. The complex base-band received signal, in this case, is given by

$$y_k = A\, e^{j\tau} s_k + n_k \tag{3}$$

where n_k is a sample of zero-mean complex Gaussian noise with variance $N_o/2$, *A* and τ are random parameters due to fading. These parameters are statistically independent and characterize the slow fading channel. The parameter *A* has a Rayliegh distribution while the parameter τ has a uniform distribution in the interval $(-\pi, \pi)$ [7, p. 529].

Let $\mathbf{s} = (s_1, s_2, ..., s_N)$ be a complex received vector of length *N* that contains the complex modulation sequence of MPSK signal, then the received sequence in a vector form is expressed as

$$\mathbf{y} = A e^{j\tau}\, \mathbf{s} + \mathbf{n} \tag{4}$$

where **n** is a zero-mean independent and identically distributed (i.i.d), complex, Gaussian noise vector. To compute the E-step of the algorithm, we need to specify the incomplete and the complete data mentioned above. The incomplete data is the observation vector **y** whose components are defined by (4), and the complete data is $\mathbf{x}=(\mathbf{y},A,\tau)$. The likelihood function (LF) of the incomplete data **y**, given the random parameters $\tau, A,$ and $\mathbf{s}$, and normalized to the power density function of the noise, is given by [8]

$$\Delta[\mathbf{y}/\tau,A,\mathbf{s}]=\frac{\dfrac{1}{(\pi N_o)^N}\exp\left\{\dfrac{-\left\|\mathbf{y}-Ae^{j\tau}\mathbf{s}\right\|^2}{N_o}\right\}}{\dfrac{1}{(\pi N_o)^N}\exp\left\{\dfrac{-\left\|\mathbf{y}\right\|^2}{N_o}\right\}} \tag{5}$$

where $\left\|\mathbf{y}-Ae^{j\tau}\mathbf{s}\right\|^2=\sum_{n=0}^{N-1}\left|y_{k-n}-As_{k-n}e^{j\tau}\right|^2$. Simplifying the LF yields

$$\Delta[\mathbf{y}/\tau,A,\mathbf{s}]=\exp\left\{-\frac{1}{N_o}\left(\sum_{n=0}^{N-1}A^2\left|s_{k-n}\right|^2-2A\left|\sum_{n=0}^{N-1}y_{k-n}^{*}\,s_{k-n}\right|\cos\left[\tau-\theta(y_{k-n}^{*},s_{k-n})\right]\right)\right\} \tag{6}$$

where * denotes the complex conjugate, $\theta(y_{k-n}^{*},s_{k-n})$ is the angle of y_{k-n}^{*},s_{k-n} assuming that the symbol s_{k-n} was sent. Note that for MPSK, the term $\left|s_{k-n}\right|^2$ is constant for all phases and then it can be dropped from the LF without affecting its maximization. Then, the log-likelihood function can be expressed as:

$$\Delta_1[\mathbf{y}/\tau,A,\mathbf{s}]=\frac{2}{N_o}A\left|\sum_{n=0}^{N-1}y_{k-n}^{*}\,s_{k-n}\right|\cos\left[\tau-\theta(y_{k-n}^{*},s_{k-n})\right] \tag{7}$$

At the *i*-th iteration, the E-step of the EM algorithm is given by:

$$U(\mathbf{s}/\mathbf{s}^{(i)})=\frac{2}{N_o}\left|\sum_{n=0}^{N-1}y_{k-n}^{*}\,s_{k-n}\right|E\left(A\cos\left[\tau-\theta(y_{k-n}^{*},s_{k-n})\right]/\mathbf{y},\mathbf{s}^{(i)}\right) \tag{8}$$

where $\mathbf{s}^{(i)}$ is the most recent sequence estimate at the *i*-th iteration of the EM algorithm. The conditional expectation in (8) is with respect to the conditional density of the random parameters *A* and τ given the incomplete data **y** and assuming that $\mathbf{s}=\mathbf{s}^{(i)}$. Since the parameters *A* and τ are statistically independent, then the conditional density $p(A,\tau/\mathbf{y},\mathbf{s}^{(i)})$ can be expressed as

$$p(A,\tau/\mathbf{y},\mathbf{s}^{(i)})=c\,p(\mathbf{y}/A,\tau,\mathbf{s}^{(i)})\,p(A)\,p(\tau) \tag{9}$$

where *c* is constant and its value can be evaluated. Using (6), the conditional density

$p(\mathbf{y}/A,\tau,\mathbf{s}^{(i)})$ can be written as

$$p(\mathbf{y}/A,\tau,\mathbf{s}^{(i)})=\exp\left\{\frac{2}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right| A\cos\left[\tau-\theta(y^*_{k-n},s^{(i)}_{k-n})\right]\right\} \tag{10}$$

By substitution of $p(\mathbf{y}/A,\tau,\mathbf{s}^{(i)})$, $p(A)$ and $p(\tau)$ in (9) we have

$$p(A,\tau/\mathbf{y},\mathbf{s}^{(i)})=c\frac{A}{\pi b}\exp\left\{\frac{2}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right| A\cos\left[\tau-\theta(y^*_{k-n},s^{(i)}_{k-n})\right]\right\}e^{-A^2/b} \tag{11}$$

Now the conditional expectation in (8) can be evaluated and then $U(\mathbf{s},\mathbf{s}^{(i)})$ becomes

$$U(\mathbf{s}/\mathbf{s}^{(i)})=c\frac{2}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s_{k-n}\right| \int_0^\infty\int_{-\pi}^{\pi} A\cos(\tau-\theta(y^*_{k-n},s_{k-n}))$$

$$\exp\left\{\frac{2A}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right|\cos(\tau-\theta(y^*_{k-n},s^{(i)}_{k-n}))\right\}\frac{A}{\pi b}e^{-A^2/b}d\tau\, dA \tag{12}$$

Using the Fourier series expansion, the exponential term in (12) can be written as

$$\exp\left\{\frac{2A}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right|\cos\left[\tau-\theta(y^*_{k-n},s^{(i)}_{k-n})\right]\right\}=I_o\left[\frac{2A}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right|\right]+$$

$$2\sum_{m=1}^{\infty} I_m\left[\frac{2A}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right|\right]\cos\left[m(\tau-\theta(y^*_{k-n},s^{(i)}_{k-n}))\right] \tag{13}$$

where $I_o[.]$ is the modified Bessel function of the first kind and zero order and $I_m[.]$ is the modified Bessel function of the first kind and order *m*. Approximating the second sum in (13) to the first term (*m*=1) and evaluating the integrals and the constant *c* in (12), then $U(\mathbf{s},\mathbf{s}^{(i)})$ becomes

$$U(\mathbf{s}/\mathbf{s}^{(i)})=d\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s_{k-n}\right|\cos\left[\theta(y^*_{k-n},s^{(i)}_{k-n})-\theta(y^*_{k-n},s_{k-n})\right]$$

$$=d\sum_{n=0}^{N-1}\mathrm{Re}\left\{y^*_{k-n}\,s_{k-n}\exp\left\{-j\theta(y^*_{k-n},s^{(i)}_{k-n})\right\}\right\} \tag{14}$$

where $d=b\frac{2}{N_o}\left|\sum_{n=0}^{N-1} y^*_{k-n}\ s^{(i)}_{k-n}\right|$. The term *d* is a +ve constant and is independent on the modulation sequence and then it has no effect on the maximization step of the EM algorithm. Note that maximizing $U(\mathbf{s},\mathbf{s}^{(i)})$ with respect to the sequence **s** is equivalent to maximizing each symbol in the sum i.e. making symbol by symbol decision. Then, the M-step of the EM algorithm is given as follows: Compute for $n = 0,1,\ldots,N$

$$s_{k-n}^{(i+1)} = \arg \max_{s_{k-n}} \operatorname{Re}\left\{ y_{k-n}^{*}\, s_{k-n} \exp\left\{ -j\theta(y_{k-n}^{*}, s_{k-n}^{(i)}) \right\} \right\} \tag{15}$$

The algorithm can be started, at i=0, by setting $\theta(y_{k-n}^{*}, s_{k-n}^{(i)}) = 0$ in (14) and then computing $s_{k-n}^{(i+1)}$ using (15). Then at the next iteration, $\theta(y_{k-n}^{*}, s_{k-n}^{(i)})$ (the angle of $y_{k-n}^{*}\, s_{k-n}^{(i)}$) is computed and the steps are repeated until the algorithm converges. This algorithm can be practically used in MPSK receivers to recover the received symbols.

As a comparison with the optimum ML solution for this problem, it is found that: the developed algorithm iteratively maximizing the LF which enables algorithms such as Viterbi algorithm to be used. Also, the developed algorithm reduces the number of computations required by the ML approach and then reduces its complexity.

IV. Computer Simulation and Results

Experimental evaluation of the performance of the estimation algorithm is performed. The performance is measured in terms of the symbol error probability (PSE). A sequence of length N = 40 symbols of 2-PSK, QPSK and 8-PSK signals is estimated using the algorithm derived above. Once the algorithm converges, all symbols are checked for errors and the PSE is evaluated. The PSE versus the signal to noise ratio is plotted in Fig. 1. This figure shows that as the signal to noise ratio increases the PSE decreases. Simulation results indicate that the algorithm converges within two or three iterations for practical values of signal to noise ratios, which makes the algorithm to be practically implemented.

V. Conclusion

The EM algorithm is applied to the problem of modulation sequence estimation of M-ary PSK signals traveled over fading channel. It is found that the EM algorithm converges fast and has an efficient implementation that makes it of practical as well as theoretical interest. This is because it enables us to produce ML estimator of long sequences, a task that was previously computationally too complex.

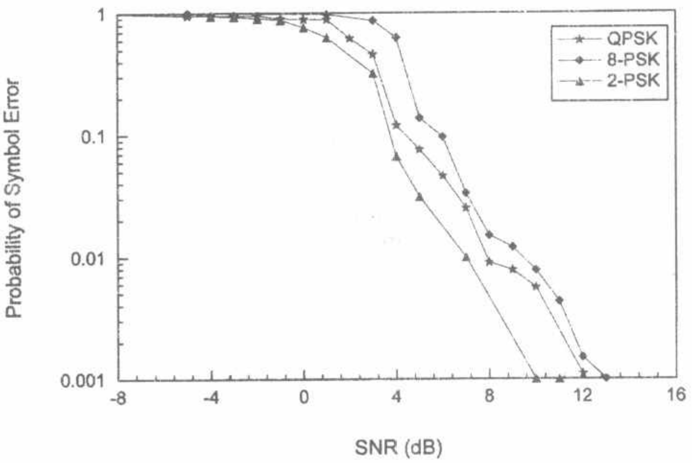

Fig. 1 : The Probability of Symbol Error Versus the Signal to Noise Ratio for 2-PSK, QPSK and 8-PSK Signals.

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