Proceeding of the 1st ICEENG-98 Conference, 24-26 March., 1998

CS3-18

MILITARY TECHNICAL COLLEGE
CAIRO - EGYPT

ICEENG 98

FIRST INTERNATIONAL CONFERENCE
ON ELECTRICAL ENGINEERING

GLQG Controller Design for the Polynomial Systems

Gamal A. El-Sheikh*

Abstract:

The solution of polynomial controller design is usually reduced to certain polynomial operations. However, these operations are given in an abstract form without clear mathematical reasoning. Therefore, this paper is devoted to present a novel derivation for the problem of polynomial generalized-linear-quadratic-gaussian (GLQG) control following a systematic approach for the derivation and considering a more general plant-structure that contains colored input disturbance and measurement noise. The presentation of the theory comes in a more concise, clear and general form to help those looking to use it without any details as well as those looking for detailed understanding and tailoring the theory to their problems. The cost function includes dynamic weighting elements allowing integral action to be introduced and robustness characteristics to be modified. Thus, the novelty of the paper stems from the fact that it presents the proof in a novel approach for a general plant structure which covers any special case in reality. The paper is supplemented with design steps and two numerical examples: one is a continuous time system and the other is a discrete time system.

Keywords: Optimal Control, Polynomial Techniques.

1- Introduction

Optimal control is a fascinating field for research in robust control design as well as the self-tuning control. In addition, it can be practically used in industrial control systems. However, it is a more mathematically complex subject compared to other methods of controller design. In addition, it is also difficult for a newcomer to get a good grasp of the field which is usually presented in a way that requires a very high level of expertise for the user. Optimal control is a well-established branch of control theory that is concerned with obtaining the best performance, in some sense, from a system. It is usually consists of a definition of the system model structure and a performance criterion. The control law is then obtained as the solution that minimizes the specified criterion, within the admissible set of control signals. Thus, the optimal control techniques will be successfully used in cases where the choice of cost function clearly and meaningfully reflected in the resultant closed-loop performance characteristics. The optimal control problem may be formulated using either state-space approach or using polynomial approach which originated by Kucera, 1979, [17] which based upon input-output models. The polynomial approach has the advantage of being straightforward for constructing the numerical algorithms and the influence of dynamic weighting on the controller is transparent. This approach, the objective of this paper, is based upon polynomial spectral factorization and diophantine equation's solution. However, the diophantine equations and the spectral factorizations are usually introduced or assumed in the theory or in its proof. Nothing mentioned upon what idea behind or why these assumptions had been used [5-15]. The reason for making it problematic and can not be understood well.

* Lecturer (B.Sc., M.Sc., Ph.D., MIEEE) in the Guidance Department, Military Technical College, Cairo, Egypt

Therefore, one objective is to give the theory in a more concise, clear and general form. Concise to help those looking to use the theory for their problem, and a clear and systematic derivation to help those looking for detailed understanding and tailoring the theory to their problems. Since the intention is usually to have an applicable theory, it should be general enough to cover most of the special cases that might exist in reality. To this end, the three theories of LQG, generalized-linear-quadratic-gaussian (GLQG) and the generalized H_∞ (GH_∞) optimal control design polynomial approach are rederived, in four parts, covering all of the above comments. The main objectives of research in this area should be concerned with generalizing and extending the previous results such that it can be applied to a wide range of industrial processes and gives optimal rejection of measurable load disturbances. In addition, the cost function weights should be dynamical (frequency-dependent) to allow various performance characteristics (including integral action) to be easily introduced.

The objective of this paper is to present a novel derivation for the problem of polynomial generalize-linear-quadratic-gaussian (GLQG) following a systematic approach for the derivation of polynomial optimal control design. The solution of polynomial approaches for controller design reduces to algorithms for polynomial stable-unstable factorizations, spectral factorizations and diophantine equations. Usually the plant structure used in most of the pertinent literature is not sufficiently general to cover most of the industrial control problems. Therefore, the other objective for the paper is to develop an GLQG-based control law for this general system structure that contains colored input disturbance and measurement noise. The cost function includes dynamic weighting elements allowing integral action to be introduced and robustness characteristics to be modified. The system model involves a reference signal filtered by a reference model, filtered disturbance corrupting the controlled output and colored measurement noise. Thus, the novelty of the paper stems from the fact that it presents the proof in a novel approach for a general plant structure which covers and could be broken down easily to any special case that might exist in reality.

Whatever the application, the jacketing software might be specialized for the case study in hand or might be general to deal with any special case and any application. Theories should be supplemented with numerical algorithms for the solution of the problem and consequently a good simulation package is indispensable. The solution of polynomial approaches for controller design reduces to algorithms for polynomial stable-unstable factorizations, spectral factorizations and diophantine equations. The results for the numerical examples in this paper are obtained using a simulation package "GAMTBX", which has been developed totally by the author in C-language as well as MATLAB routines.

The paper is organized as follows: Section-2 presents the system structure, Section-3 presents briefly the diophantine equations' solution and the spectral factorizations, Section-4 defines the control problem and motivates the controller design, Section-5 is devoted for the derivation of the GLQG controller design and the properties of the solution in addition to approaches for simplifying the solution or breaking down to special cases, and Section-6 presents the design steps and two numerical examples and finally a conclusion for the paper is given in Section-7.

2- System Description

The system is represented in the form of feedback configuration as shown in Fig. 1. In this system the high-frequency (HF) disturbance $n(t)$ affects the observed system output and the requirement is to control lower frequency (LF) variations in the output, represented by the signal $y(t)$. The signal $n(t)$ can also denote measurement noise such as tachogenerator ripple, gyroscopic errors,...etc. Whatever the type of *n(t)*, the problem is to ensure that the signal *y(t)* {rather than *z(t)*} follows the reference signal *r(t)*.

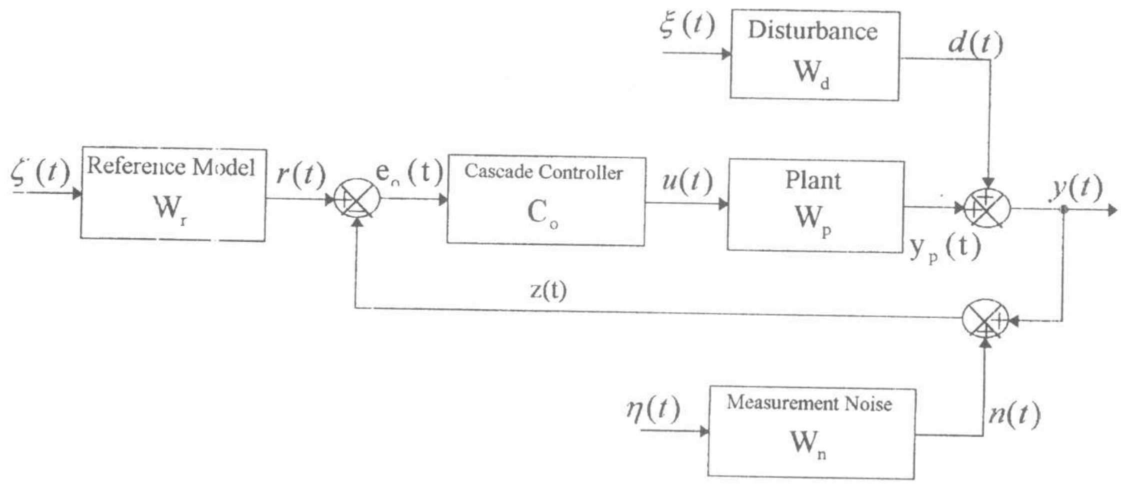

Fig. 1: Feedback control system with input disturbance, measurement noise and reference

The system polynomials (constituting the transfer functions) are considered functions of the unit-delay operator z^{-1} and the disturbance and noise sources are assumed to be mutually independent with zero mean and unit variance. The various subsystems can be represented by the coprime polynomials [4,5,13,16,17] as follows: the Plant $W_p = A^{-1}B$, input disturbance $W_d = A_d^{-1}C_d$, the output disturbance or measurement noise $W_n = A_n^{-1}C_n$, the reference $W_r = A_r^{-1}E_r$, and the controller $C_o = C_{od}^{-1}C_{on}$. The signals $\{\zeta(t), \xi(t), \eta(t)\}$ are white noise signals with zero means and unity variances. The plant polynomials $B(z^{-1})$ and $A(z^{-1})$ are free of unstable hidden modes [16,19]. The external signal-filters (W_r, W_d, W_n) are assumed to be asymptotically stable and all the denominator polynomials are distinct i.e. $A \neq A_n \neq A_d \neq A_r$. The various subsystems are assumed to be causal i.e. $A(0) = A_n(0) = A_d(0) = A_r(0) = 1$. The input disturbance polynomial C_d is strictly Hurwitz [13]. The plant zero polynomial $B(z^{-1}) = z^{-k}B_k(z^{-1})$, includes a k-step delay where $k \geq 1$. The plant, for simplicity of optimization procedure, it is assumed to be free of poles on the unit circle of the Z-plane. If the plant includes unit circle poles, there are two methods for dealing with this problem; either (i) the particular poles can be moved off the unit-circle by an amount ε; ε→0 {Wiener-Hopf optimization theory}, or (ii) the unit circle terms can be dealt with directly by integrating around small semi-circular contours which avoids these poles in the evaluation of the cost function [2,3,11,17,18, 19]. This paper presents the model structure, criterion function and solution for the polynomial GLQG control problem with a detailed proof complemented by the properties of the resultant closed-loop system and some special cases.

3- Polynomial Operations

Usually, the process of designing an optimal controller based on input-output models reduces to a process of polynomial operations including, mainly, the solution of **diophantine equations** and **spectral factorization**.

1. Diophantine equation

The polynomial equation A X + B Y = C, in which the polynomials A, B, and C are known and required a unique solution {X, Y} is called **diophantine equation**. Clearly, since there are two unknowns and only one equation, then there is an infinite number of solutions {X, Y}. So to obtain a unique solution, an arbitrary **additional constraint** should be applied or defined which is clear from the following theorem.

Theorem 1 [17]: The polynomial equation

$$AX + BY = C \tag{1}$$

has a solution {X, Y} iff the greatest common factor of A and B is also a factor of C. Thus, to obtain a unique solution {X, Y} for the above polynomial (if exists) the additional constraint is that X{Y} should be of minimal degree.

2. Spectral factorization:

It is the process of finding a stable polynomial which in some way corresponds to another polynomial that may be unstable. That is given a polynomial $X(z^{-1})$ which may or may not be stable, then the spectral factorization obtains a stable spectral factor $S(z^{-1})$ of $X(z^{-1})$which satisfies the relation:

$$S(z^{-1})S^{*}(z^{-1}) = X(z^{-1})X^{*}(z^{-1}) \quad (2)$$

Theorem 2: [17,22]

Let the polynomial $X(z^{-1})$ in Eqn(2) have no zeros on the unit circle. Then there exists a unique polynomial $S(z^{-1})$with all its zeros inside the unit circle.

The effect of spectral factorization is to reflect the unstable zeros of a polynomial about the unit circle and to leave the stable zeros where they are without any change. Therefore, spectral factorization can be made by finding all the zeros of the given polynomial and reflecting the unstable ones about the unit circle. Then, the stable spectral factor can be obtained as a combination of the stable zeros and the reflections of the unstable ones.

4- Problem Definition and Optimal Solution

The optimal control problem requires the definition of a control law structure and the cost function to be minimized. The optimal control law for the stochastic tracking problem to be defined in **Problem-1** is given in **Theorem-3**. The solution involves polynomial spectral factorization and solution of the diophantine equations, with a detailed proof.

4.1 Control structure

The controller structure given in Fig. 1 is known as a Single-Degree-Of-Freedom (SDOF) controller since the reference and observed output signal have the same weight to yield the control signal *u(t)*. This controller might be replaced by a controller with separate reference *r(t)* and observation *z(t)* inputs. That is, there will be two controllers, respectively, reference controller and feedback or cascade controller which enable a smaller cost-function value to be achieved and good command following. However, it is common in industrial process control to use an integrator in the error channel for reducing the steady state error. This structure is called Two-Degree-Of-Freedom (TDOF) and it is out of the paper objective.

Stability Lemma: (Kucera 1979) [17]

The closed loop system with both plant and controller free of unstable hidden modes is asymptotically stable iff the optimal controller satisfy

$$AC_{od} + BC_{on} = 1 \quad (3)$$

where the control law can be written as $u(t) = C_o e_o(t)$ and the cascade controller is defined as $C_o = \frac{C_{on}(z^{-1})}{C_{od}(z^{-1})}$.

4.2 Problem-1:

The GLQG performance criterion to be minimized by the control law defined above is given by

$$J = E\left[\psi^2(t)\right] = \frac{1}{2\pi j}\oint_{|z|=1} \phi_{\psi\psi}(z^{-1})\frac{dz}{z} \quad (4)$$

$$\psi = P_c \cdot e(t) + F_c \cdot u(t) \quad (5)$$

Where, ψ is a weighted signal composed of the tracking error *e(t)* and the control signal *u(t)* with the weighting elements P_c and F_c as rational transfer functions. The $\phi_{\psi\psi}(z^{-1})$ represents the spectral density of the weighted signal. The weighting elements P_c and F_c are defined as

$$P_c = P_{cd}^{-1}P_{cn} \quad \& \quad F_c = F_{cd}^{-1}F_{cn} \quad (6)$$

Where, P_{cd} and F_{cd} are strictly Schur polynomials with $P_{cd}(0) = F_{cd}(0) = 1$ and F_{cn} might be given with delay. Let us define the polynomial L, to be used later in the derivation, as follows:

$$L = L_1 L_2 = P_{cn} F_{cd} B - F_{cn} P_{cd} A \tag{7}$$

then, the weighting elements are chosen to ensure that *L* is free of zeros on the unit-circle of the Z-Plane, or $(L^* L)$ is positive definite on $|z|=1$. Those definitions ensure that the spectral factors D_c and D_f are strictly Schur.

5- GLQG Controller Design with General Processes

5.1 Theorem 3: GLQG Optimal control law

The GLQG optimal control, which is a solution to Problem-1, is given such that the controller transfer function is given by

$$C_o = \frac{GF_{cd}A - (XA_d A_r F_{cd})A}{HP_{cd}A_r + (XA_d A_r F_{cd})B} \tag{8}$$

Which requires the solution of two spectral factorizations and three diophantine equations as follows:

1. **Spectral factors (D_c, D_f):**

The strictly schur spectral factors D_c and D_f which will be used in solving the diophantine equations are defined as follows:

$$D_c^* D_c = (P_{cn}F_{cd}B - F_{cn}P_{cd}A)^*(P_{cn}F_{cd}B - F_{cn}P_{cd}A) \tag{9-a}$$

$$D_f^* D_f = E_r^* E_r A_d^* A_d A_n^* A_n + C_d^* C_d A_r^* A_r A_n^* A_n + C_n^* C_n A_r^* A_r A_d^* A_d \tag{9-b}$$

2. **The diophantine equations**:

The polynomials G and H are obtained from the minimal degree solution {G, H, F} w.r.t. F of the coupled diophantine equations;

$$FA_d A_n A_r P_{cd} + L_2 G = P_{cn} D_f \tag{10-a}$$

$$FBA_d A_n A_r F_{cd} - L_2 HA_r = F_{cn} D_f A \tag{10-b}$$

and {X, Y} the minimal degree solution w.r.t. Y of the diophantine equation;

$$YA_n P_{cd} + D_{fc}^* z^{-g_1} X = P_{cn} C_n C_n^* A_d^* A_r^* L^* z^{-g_1} \tag{11}$$

Where;

$$g_1 = \max(n_{D_{fc}}, n_{C_n} + n_{A_d} + n_{A_r} + n_A + n_L) \tag{12-a}$$

$$D_{fc} = D_f D_c \tag{12-b}$$

5.2 Proof of Theorem-3:

The proof of **Theorem 3** will be carried out through three steps of algebraic manipulation of the cost function augmented by a step for minimizing this cost as follows;

5.2.1 First step:

The first step is to separate the cost function into two parts, one is depending on the controller C_o and the other is not. From the system model equations, the control signal *u(t)* can be given as follows:

$$\begin{aligned} u(t) &= C_o\{r(t) - y(t) - n(t)\} \\ &= C_o\{r(t) - W_p u(t) - d(t) - n(t)\} \\ &= C_o r(t) - W_p C_o u(t) - C_o d(t) - C_o n(t) \end{aligned} \tag{13}$$

which implies that

$$\begin{aligned} u(t) &= \frac{C_o}{1+W_p C_o}\{r(t) - d(t) - n(t)\} \\ &= M\{r(t) - d(t) - n(t)\} \end{aligned} \tag{14}$$

Where, M is the control sensitivity function [1,4,20,23] that is given as:

$$M = \frac{C_o}{1+W_p C_o} = C_o S \tag{15}$$

where S is the sensitivity function defined [1,4,20,23] as follows:

$$S = \frac{1}{1+W_pC_o} = 1 - W_pM \tag{16}$$

The system output *y(t)* is given by

$$\begin{aligned} y(t) &= W_pu(t) + d(t) \\ &= W_pM\{r(t)-n(t)\} + [1-W_pM]d(t) \end{aligned} \tag{17}$$

Then the tracking error *e(t)* can be given by

$$\begin{aligned} e(t) &= r(t) - y(t) \\ &= [1-W_pM]\{r(t)-d(t)\} + W_pM\, n(t) \end{aligned} \tag{18}$$

According to the stability theorem, the following relations can be obtained:

$$C_o = \frac{C_{on}}{C_{od}} \tag{19-a}$$

$$S = \frac{1}{1+W_pC_o} = \frac{AC_{od}}{AC_{od}+BC_{on}} = AC_{od} \tag{19-b}$$

$$T = \frac{W_pC_o}{1+W_pC_o} = \frac{BC_{on}}{AC_{od}+BC_{on}} = BC_{on} \tag{19-c}$$

$$M = \frac{C_o}{1+W_pC_o} = \frac{AC_{on}}{AC_{od}+BC_{on}} = AC_{on} \tag{19-d}$$

where S, T and M are, respectively, the sensitivity, complementary sensitivity and control sensitivity functions. Substituting these equations (19) into the tracking error (18) and control signal (14) yields:

$$e(t) = (1-BC_{on})\{r(t)-d(t)\} + BC_{on}n(t) \tag{20-a}$$

$$u(t) = AC_{on}\{r(t)-d(t)-n(t)\} \tag{20-b}$$

The weighted signal ψ(t) can be obtained as follows:

$$\begin{aligned} \psi(t) &= P_c \cdot e(t) + F_c \cdot u(t) \\ &= P_c \cdot (1-BC_{on})\{r(t)-d(t)\} + P_cBC_{on}n(t) + F_cAC_{on}\{r(t)-d(t)-n(t)\} \\ &= [P_c \cdot (1-BC_{on}) + F_cAC_{on}]\{r(t)-d(t)\} + [P_cB - F_cA]C_{on}n(t) \end{aligned} \tag{21}$$

Therefore, the integrand of the cost function or the spectral density of the weighted signal ψ(t) can be obtained as follows:

$$\begin{aligned} \phi_{\psi\psi}(t) &= [P_c \cdot (1-BC_{on}) + F_cAC_{on}]^* \phi_o [P_c \cdot (1-BC_{on}) + F_cAC_{on}] \\ &\quad + C_{on}^*[P_cB - F_cA]^* \phi_{nn} [P_cB - F_cA]C_{on} \\ &= [P_c^*P_c - P_c^*(P_cB - F_cA)C_{on} - C_{on}^*(P_cB - F_cA)^*P_c]\phi_o \\ &\quad + [C_{on}^*(P_cB - F_cA)^*(P_cB - F_cA)C_{on}](\phi_o + \phi_{nn}) \end{aligned} \tag{22}$$

where the spectral density ϕ_o is defined as the sum of spectral densities of the reference and disturbance signals as follows

$$\phi_o = \phi_{rr} + \phi_{dd} \tag{23}$$

To simplify the integrand of the cost function, Eqn(22), the **Control spectral factor** may be defined as

$$\begin{aligned} Y_c^*Y_c = \frac{D_c^*D_c}{A_c^*A_c} &= (P_cB - F_cA)^*(P_cB - F_cA) \\ &= \frac{(P_{cn}F_{cd}B - F_{cn}P_{cd}A)^*(P_{cn}F_{cd}B - F_{cn}P_{cd}A)}{(P_{cd}F_{cd})^*(P_{cd}F_{cd})} \end{aligned} \tag{24}$$

Therefore,

$$D_c^* D_c = (P_{cn} F_{cd} B - F_{cn} P_{cd} A)^* (P_{cn} F_{cd} B - F_{cn} P_{cd} A) \qquad (25\text{-a})$$

$$A_c = (P_{cd} F_{cd}) \qquad (25\text{-b})$$

$$Y_c = \frac{D_c}{(P_{cd} F_{cd})} \qquad (25\text{-c})$$

And the **Filter spectral factor** may be defined as

$$Y_f^* Y_f = \frac{D_f^* D_f}{A_f^* A_f} = \phi_T + \phi_{dd} + \phi_{nr} = \phi_o + \phi_{nn} \qquad (26\text{-a})$$

$$= \frac{E_r^* E_r}{A_r^* A_r} + \frac{C_d^* C_d}{A_d^* A_d} + \frac{C_n^* C_n}{A_n^* A_n}$$

$$= \frac{E_r^* E_r A_d^* A_d A_n^* A_n + C_d^* C_d A_r^* A_r A_n^* A_n + C_n^* C_n A_r^* A_r A_d^* A_d}{(A_d A_n A_r)^* (A_d A_n A_r)} \qquad (26\text{-b})$$

Therefore,

$$D_f^* D_f = E_r^* E_r A_d^* A_d A_n^* A_n + C_d^* C_d A_r^* A_r A_n^* A_n + C_n^* C_n A_r^* A_r A_d^* A_d \qquad (27\text{-a})$$

$$A_f = A_d A_n A_r \qquad (27\text{-b})$$

$$Y_f = \frac{D_f}{(A_d A_n A_r)} \qquad (27\text{-c})$$

Substituting eqns(24,26) into eqns(22) yields:

$$\phi_{\psi\psi}(t) = [P_c^* P_c - P_c^* (P_c B - F_c A) C_{on} - C_{on}^* (P_c B - F_c A)^* P_c] \phi_o + C_{on}^* Y_c^* Y_c C_{on} Y_f^* Y_f \qquad (28)$$

Let us define the following auxiliary spectra, that may simplify the integrand, as follows:

$$\phi_h = P_c (P_c B - F_c A)^* \phi_o \qquad (29\text{-a})$$

$$\phi_h^* = P_c^* (P_c B - F_c A) \phi_o \qquad (29\text{-b})$$

Thus, the integrand of the cost function, Eqn(28), can then be written as follows:

$$\phi_{\psi\psi}(t) = P_c^* P_c \phi_o - \phi_h^* C_{on} - C_{on}^* \phi_h + C_{on}^* Y_c^* Y_c C_{on} Y_f^* Y_f \qquad (30)$$

Now, it is clear that the cost function had been separated into parts depending on the controller C_o and others are not.

5.2.2 Second step:

The second step in the proof is to split each term of the integrand, Eqn (30), more into parts which depend on the controller, and parts which do not depend on it using what is called **Completing Squares** [17]. Considering the identity;

$$XX^* = (X - A)(X - A)^* - AA^* + AX^* + A^* X \qquad (31)$$

and completing the squares in Eqn(30) yield

$$\phi_{\psi\psi} = \left(Y_c C_{on} Y_f - \frac{\phi_h}{Y_c^* Y_f^*} \right) \left(Y_c C_{on} Y_f - \frac{\phi_h}{Y_c^* Y_f^*} \right)^* + P_c^* P_c \phi_o - \frac{\phi_h \phi_h^*}{(Y_c^* Y_f^*)(Y_c Y_f)} \qquad (32)$$

Notice that the final two terms do not depend on the controller and, therefore, will not enter into the subsequent minimization argument.

5.2.3 Third step:

The third step in the proof is to split the C_o-dependent terms of Eqn(32) into its **stable and unstable** parts. According to the definitions of different subsystems and Eqns(6, 29) the following can be obtained

$$\frac{\phi_h}{Y_c^* Y_f^*} = \frac{\phi_o (P_c B - F_c A)^* P_c}{Y_c^* Y_f^*} = \frac{(Y_f^* Y_f - \phi_{nn})(P_c B - F_c A)^* P_c}{Y_c^* Y_f^*} = \frac{D_f (P_{cn} F_{cd} B - F_{cn} P_{cd} A)^* P_{cn}}{A_d A_n A_r P_{cd} D_c^*} - \frac{C_n^* C_n A_d^* A_r^* (P_{cn} F_{cd} B - F_{cn} P_{cd} A)^* P_{cn}}{A_n P_{cd} D_c^* D_f^*} \quad (33)$$

Since the terms $\{A_d A_n A_r P_{cd}, A_n P_{cd}\}$ are stable and $\{D_c^*, D_{fc}^*\}$ are unstable (because D_c and D_f are strictly schur spectral factors), the two parts of Eqn(33) can be splitted into **stable and unstable** parts by introducing the polynomials $\{G, F\}$ and $\{X, Y\}$ as follows:

$$\frac{\phi_h}{Y_c^* Y_f^*} = \left[\frac{G}{A_d A_n A_r P_{cd}} + \frac{F z^g}{D_c^*}\right] - \left[\frac{X}{A_n P_{cd}} + \frac{Y z^{g_1}}{D_{fc}^*}\right] = \left[\frac{D_c^* G + A_d A_n A_r P_{cd} F z^g}{A_d A_n A_r P_{cd} D_c^*}\right] - \left[\frac{D_{fc}^* X + A_n P_{cd} Y z^{g_1}}{A_n P_{cd} D_{fc}^*}\right] \quad (34)$$

Comparing Eqns(33, 34) yields, respectively, the **first** and **third** diophantine equations as follows:

$$F A_d A_n A_r P_{cd} + D_c^* G z^{-g} = P_{cn} D_f (P_{cn} F_{cd} B - F_{cn} P_{cd} A)^* z^{-g} \quad (35\text{-a})$$

$$Y A_n P_{cd} + D_{fc}^* X z^{-g_1} = P_{cn} C_n^* C_n A_d^* A_r^* (P_{cn} F_{cd} B - F_{cn} P_{cd} A)^* z^{-g_1} \quad (35\text{-b})$$

According to Eqns(25,27,33), the squared term in Eqn(32) becomes as follows:

$$Y_c C_{on} Y_f - \frac{\phi_h}{Y_c^* Y_f^*} = \frac{D_c D_f C_{on}}{A_d A_n A_r F_{cd} P_{cd}} - \left[\frac{G}{A_d A_n A_r P_{cd}} + \frac{F z^g}{D_c^*}\right] + \left[\frac{X}{A_n P_{cd}} + \frac{Y z^{g_1}}{D_{fc}^*}\right]$$

$$= \frac{D_c D_f C_{on} - G F_{cd} + X A_d A_r F_{cd}}{A_d A_n A_r F_{cd} P_{cd}} - \left[\frac{F z^g}{D_c^*} - \frac{Y z^{g_1}}{D_{fc}^*}\right] \quad (36\text{-a})$$

$$= T_1^- + T_1^+ \quad (36\text{-b})$$

where;

$$T_1^- = \frac{D_c D_f C_{on} - G F_{cd} + (X A_d A_r F_{cd})}{A_d A_n A_r F_{cd} P_{cd}} \quad (37\text{-a})$$

$$T_1^+ = \left[\frac{F z^g}{D_c^*} - \frac{Y z^{g_1}}{D_{fc}^*}\right] \quad (37\text{-b})$$

Now the M dependent term of the integrand had been splitted into a **stable** term T_1^- and an **unstable** term T_1^+. The denominator of T_1^- is stable since $A_d, A_n, A_r, P_{cd}, F_{cd} \in P_-(z^{-1}) \cup P_0(z^{-1})$ while the denominator of T_1^+ is strictly unstable since $D_c \in P_-(z^{-1})$ and $\overline{D}_c \in P_+(z^{-1})$, according to its definition. Note that; $P_-(z^{-1})$ represents the interior of the unit circle, $P_0(z^{-1})$ represents the boundary of the unit circle and $P_+(z^{-1})$ represents the exterior of the unit circle.

5.2.4 Minimization:

Substituting Eqn(37) into Eqns(32) yield

$$\phi_{\psi\psi} = [T_1^- + T_1^+][T_1^- + T_1^+]^* + P_c^* P_c \phi_o - \frac{\phi_h \phi_h^*}{(Y_c^* Y_f^*)(Y_c Y_f)}$$

$$= [T_1^- T_1^{-*} + T_1^- T_1^{+*} + T_1^+ T_1^{-*} + T_1^+ T_1^{+*}] + P_c^* P_c \phi_o - \frac{\phi_h \phi_h^*}{(Y_c^* Y_f^*)(Y_c Y_f)} \quad (38)$$

In Eqn(38), T_1^- represent stable term while T_1^+ represent the unstable term. The terms $T_1^- T_1^{+*}$ are analytic within the unit-disc $|z| \le 1$ of the Z-plane [2,16]. Thus, using the residue theorem, the integrals of the cross terms in Eqn(38) are zero i.e.

$$\oint_{|z|=1} \left\{T_1^- T_1^{+*}\right\} \frac{dz}{z} = \oint_{|z|=1} \left\{T_1^+ T_1^{-*}\right\} \frac{dz}{z} = 0 \tag{39}$$

Therefore the cost function can be simplified to

$$J = \frac{1}{2\pi j} \oint_{|z|=1} \left\{ T_1^- T_1^{-*} + T_1^+ T_1^{+*} + P_c^* P_c \phi_o - \frac{\phi_h \phi_h^*}{(Y_c^* Y_f^*)(Y_c Y_f)} \right\} \frac{dz}{z} \tag{40}$$

Since the terms T_1^+ and the last two terms are independent of the controller, the criterion **J** can be minimized by setting $T_1^- = 0$. The idea behind splitting the terms of the criterion integrand into stable and unstable parts is that the integral of the cross parts will be zero (by the Residue theorem) [2,4,16,21] and the integrals of the stable parts can be set to zero by appropriate choice of the controller transfer function. Therefore, **Setting** $T_1^- = 0$ into Eqns(37) yields

$$C_{on} = \frac{GF_{cd} - (XA_d A_r F_{cd})}{D_c D_f} = \frac{GF_{cd} A - (XA_d A_r F_{cd})A}{D_c D_f A} \tag{41}$$

Using Eqn(41) and the stability Lemma yield the controller denominator polynomial as follows:

$$C_{od} = \frac{1 - BC_{on}}{A} = \frac{D_c D_f - GF_{cd} B + (XA_d A_r F_{cd})B}{D_c D_f A} \tag{42}$$

Therefore, the minimizing feedback controller C_o becomes

$$C_o = \frac{C_{on}}{C_{od}} = \frac{GF_{cd} A - (XA_d A_r F_{cd})A}{D_c D_f - GF_{cd} B + (XA_d A_r F_{cd})B} \tag{43}$$

For more convenient solution (form of the controller transfer function), consider the equation

$$HP_{cd} A_r = D_c D_f - GF_{cd} B \tag{44-a}$$

$$\Rightarrow GF_{cd} B + HP_{cd} A_r = D_c D_f \tag{44-b}$$

Multiplying Eqn(35-a) by $\{F_{cd} B\}$ and Eqn(44-b) by $\{D_c^* z^{-g}\}$, then subtracting and using the spectral factorization (25-a) yield the **second diophantine** equation

$$FBA_d A_n A_r F_{cd} - D_c^* z^{-g} HA_r = F_{cn} D_f A (P_{cn} F_{cd} B - F_{cn} P_{cd} A)^* z^{-g} \tag{45}$$

Accordingly, the feedback controller C_o becomes

$$C_o = \frac{C_{on}}{C_{od}} = \frac{GF_{cd} A - (XA_d A_r F_{cd})A}{HP_{cd} A_r + (XA_d A_r F_{cd})B} \tag{46}$$

Considering Eqns(25,26,29), the last two terms in Eqn(40) cancel each other as follows:

$$\begin{aligned} P_c^* P_c \phi_o - \frac{\phi_h \phi_h^*}{(Y_c^* Y_f^*)(Y_c Y_f)} &= P_c^* P_c \phi_o - \frac{P_c (P_c B - F_c A)^* \phi_o P_c^* (P_c B - F_c A)\phi_o}{(Y_c^* Y_f^*)(Y_c Y_f)} \\ &= P_c^* P_c \phi_o - \frac{P_c Y_c^* Y_f^* Y_c Y_f \phi_o P_c^*}{(Y_c^* Y_f^*)(Y_c Y_f)} \\ &= 0 \end{aligned} \tag{47}$$

Therefore, the obtained solution gives minimum cost function J_{min} as

$$J_{min} = \frac{1}{2\pi j} \oint_{|z|=1} \left\{T_1^+ T_1^{+*}\right\} \frac{dz}{z} \tag{48}$$

Considering the definition of L given in Eqn(7) and its factorization into $L_1 L_2$; the following relations hold

$$\bar{L} = \bar{L}_1 \bar{L}_2 \tag{49-a}$$

$$D_c^* D_c = L_1^* L_2^* L_1 L_2 \tag{49-b}$$

$$D_c = L_1 \bar{L}_2 \tag{49-c}$$

$$\bar{D}_c = \bar{L}_1 L_2 \tag{49-d}$$

Considering Eqns(7,49), the diophantine equations (35-a,45,35-b) can be put in the following form:

$$FA_dA_nA_rP_{cd} + \bar{L}_1L_2G = P_{cn}D_f\bar{L}_1\bar{L}_2 \quad (50\text{-a})$$

$$FBA_dA_nA_rF_{cd} - \bar{L}_1L_2HA_r = F_{cn}D_fA\bar{L}_1\bar{L}_2 \quad (50\text{-b})$$

$$YA_nP_{cd} + D_{fc}^*Xz^{-g_1} = P_{cn}C_n^*C_nA_d^*A_r^*L^*z^{-g_1} \quad (50\text{-c})$$

These equations can be simplified by redefining the unknown polynomials: $F \rightarrow F\bar{L}_1\bar{L}_2$, $G \rightarrow G\bar{L}_2$, and $H \rightarrow H\bar{L}_2$ to yield the diophantine equations as follows:

$$FA_dA_nA_rP_{cd} + L_2G = P_{cn}D_f \quad (51\text{-a})$$

$$FBA_dA_nA_rF_{cd} - L_2HA_r = F_{cn}D_fA \quad (51\text{-b})$$

$$YA_nP_{cd} + D_{fc}^*Xz^{-g_1} = P_{cn}C_n^*C_nA_d^*A_r^*L^*z^{-g_1} \quad (51\text{-c})$$

□

5.3 Properties of the solution

5.3.1 Implied diophntine equation:

The implied diophantine equation can be obtained by multiplying Eqn(35-a) by (BF_{cd}) and Eqn(45) by P_{cd} then substracting and using the spectral factorization (25-a) yield

$$GBF_{cd} + HA_rP_{cd} = D_cD_f \quad (52)$$

And the characteristic equation is given by

$$\rho_c = A(GBF_{cd} + HA_rP_{cd}) = AD_cD_f \quad (53)$$

This equation shows that **the system is guaranteed to be closed-loop stable** if the system/plant poles *A* is stable since both D_c and D_f are defined to be stable. The controller has poles due to the weighting terms P_{cd} and F_{cd} and zeros due to F_{cd} which demonstrates the effect of the dynamic cost function elements. For example, integral action may be easily introduced by defining P_{cd} and F_{cd} as integrators [i.e. $=1-z^{-1}$], upon which an infinite error costing at zero frequency will be introduced.

5.3.2 GLQG Controller design with guaranteed stability

Without loss of generality, the plant and the reference subsystems can be assumed to have common pole polynomials (i.e $A = A_r$) to simplify the computation of controller. Thus, the optimal GLQG controller has the following transfer function:

$$C_o = \frac{C_{on}}{C_{od}} = \frac{GF_{cd} - (XA_dF_{cd})A}{HP_{cd} + (XA_dF_{cd})B} \quad (54)$$

Which requires the solution of three diophantine equations and two spectral factorizations as obtained before, taking into consideration the present assumption. The manipulation of the diophantine equations yields the the characteristic polynomial:

$$\rho_c = AP_{cd}H + BF_{cd}G = D_cD_f \quad (55)$$

And this shows that **the system is guaranteed to be closed-loop stable,** since both D_c and D_f are defined to be stable, **without any restriction upon the open-loop stability or the non-minimum-phase property**.

Both the closed loop stability and internal stability are guaranteed since

$$\frac{HP_{cd}}{D_cD_f}A + \frac{GF_{cd}}{D_cD_f}B = M_oA + N_oB = 1 \quad (56)$$

Thus, the controller transfer function (54) can be written as

$$C_o = \frac{N_o - KA}{M_o + KB} \quad (57)$$

Where $N_o = \frac{GF_{cd}}{D_cD_f}$, $M_o = \frac{HP_{cd}}{D_cD_f}$ and $K = \frac{XA_dF_{cd}}{D_cD_f}$. It is clear that the controller satisfies the necessary and sufficient condition for stability derived by Desoer *et al.* [2,3] which represents a generalization of the results given by Youla, *et al.* [22]. It should be noted that the control law is stabilizing for arbitrary asymptotically stable *K* and hence errors in the computation of X can not result in instability. This is fortunate because X depends on the output disturbance $w_n = \frac{C_n}{A_n}$ which involves inaccuracies due to lack of knowledge and randomness.

5.3.3 GLQG Controller design without measurement noise

The controller design can be facilitated more by neglecting the measurement noise in system structure i.e. $C_n = 0$ in Fig.1. The external signal-filters (W_r, W_d) are assumed to be asymptotically stable and the plant, the reference and the input disturbance denominator polynomials are assumed equal i.e. $A = A_r = A_d$. The plant may be either open-loop stable, unstable or non-minimum phase, but for simplicity of optimization procedure, it is assumed to be free of poles on the unit circle of the Z-plane. Thus, the optimal GLQG controller for this process is given by the following transfer function:

$$C_o = \frac{C_{on}}{C_{od}} = \frac{GF_{cd}}{HP_{cd}} \qquad (58)$$

The characteristic polynomial for this case has the form

$$\rho_c = AP_{cd}H + BF_{cd}G = D_c D_f \qquad (59)$$

And this equation shows that **the system is guaranteed to be closed-loop stable,** since both D_c and D_f are defined to be stable, **without any restriction upon the open-loop stability or the non-minimum-phase property**. In addition, both the closed loop stability and internal stability can be investigated as before, where N_o and M_o have the same form as before while K=0.

6- Numerical Implementation:

6.1 Design steps

Having stated the GLQG stochastic tracking control law, the design procedure can be summarized in stepwise as follows

Data:
- Obtain the model polynomials A, B, and C_d.
- Obtain the polynomials E_r and A_r of the reference model.
- Obtain the polynomials C_d and A_d of the disturbance model.
- Obtain the polynomials C_n and A_n of the measurement noise model
- Choose the cost function weights Q_c and R_c.

Step-1 : Obtain the spectral factors
Step-2 : Solve the coupled diophantine equations for G and H.
Step-3 : Solve the diophantine equation for {X, Y}.
Step-4 : Calculate the controller using the appropriate form.

6.2 Continuous process example:

Consider a system described by the following transfer function polynomials

$$W_p = \frac{B(s)}{A(s)} = \frac{1}{0.1s^2 + 1.1s + 1} \qquad W_d = \frac{C_d(s)}{A(s)} = \frac{0.1s - 1}{0.1s^2 + 1.1s + 1}$$

Then, consider the following simple cost weights: $P_c = 1/s$ and $F_c = 1$. The design process yield the following controller transfer function: $C_o = \frac{0.1256s^2 + 1.3557s + 1}{s(0.1s + 1.1185)}$

6.3 Discrete process example:

Consider a system described by the transfer function polynomials

$$W_p = \frac{B(z^{-1})}{A(z^{-1})} = \frac{z^{-2}(1 + 2z^{-1})}{1 - 0.95z^{-1}} \qquad W_d = \frac{C_d(z^{-1})}{A(z^{-1})} = \frac{1 - 0.7z^{-1}}{1 - 0.95z^{-1}}$$

The cost weights may be chosen as $P_c = 1/(2 - z^{-1})$ and $F_c = 1$. The design process yield the following controller transfer function: $C_o = \frac{0.4243 - 0.1949z^{-1}}{3.5595 + 0.0636z^{-1} + 1.5551z^{-2} - 1.2384z^{-3}}$

7- Conclusions

This paper presented the derivation of the optimal GLQG control theory, polynomial approach, in a novel form following a systematic approach which is more concise, clear and general. The derivation was based on a general system structure which contains colored input disturbance and measurement noise. The theory is presented in a more concise, clear and general form to help those looking to use it without any details as well as those looking for detailed understanding and tailoring the theory to their problems. The cost function weights may be dynamical (frequency-dependent) to allow various performance characteristics (including integral action) to be easily introduced and robustness characteristics to be modified. The system is guaranteed to be closed-loop stable since both spectral factors D_c and D_f are defined to be stable without any restriction upon the open-loop stability or the non-minimum-phase property, which is a great advantage during controller design. This approach is applicable to both continuous and discrete time systems as clear from the numerical examples given at the end of the paper.

References:

[1] Astrom, K. J. and B. Wittenmark, *Computer Controlled Systems: Theory and Design*, Prentice-Hall, 1990.

[2] Desoer, C.A. and M. Vidyasagar, *Feedback Systems: Input-Output Properties*,Academic Press, 1975.

[3] Desoer, C.A., R.W. Liu, J. Murray and R. Saeks, *Feedback System Design: The fractional representation approach to analysis and synthesis*, IEEE Trans. Aut. Cont., AC-25, pp399-412, 1980.

[4] Doyle, J., B.A. Francis, and A.R. Tannenbaum, *Feedback Control Theory*, Macmillan Publishing Company, 1992.

[5] Fairbairn, N.A. , *Polynomial Systems LQG and H_∞ Control with Marine Applications*, Ph.D. Thesis, University of Strathclyde, Dept. of E.E.E./ICU, 1989.

[6] Fung and M.J.Grimble, *The Adaptive Tracking of Slowly Varying Processes with Coloured-Noise Disturbances*, Trans. Inst. M. C., Vol.6, No.6, Oct-Dec 1984.

[7] Grimble, M.J., *Solution of the Discrete-time Stochastic Optimal Control Problem in the Z-Domain*, Int. J. Sys. Science, 1979.

[8] Grimble, M.J., *The Design of S-Domain Optimal Controllers with Integral Action for Output Feedback Control Systems*, INT. J.C., Vol.31, No.5, pp869-882, 1980.

[9] Grimble, M.J., *Implicit and Explicit LQG Self-Tuning Controllers*, Automatica, Vol.20, No.5, pp 661-669, 1984.

[10] Grimble, M.J., *Controllers for LQG Self-Tuning Applications with Coloured Measurement noise and Dynamic Costing*, IEE Part-D, Vol.133, No.1, January 1986.

[11] Grimble, M.J. , H_∞ *Robust Controller for Self-Tuning Control Applications, Part 1: Controller Design*, INT. J. Control, 1987, Vol.46, No.4, 1429-1444.

[12] Grimble, M.J. , H_∞ *Robust Controller for Self-Tuning Control Applications, Part 2: Self-Tuning and Robustness*, INT. J. Control, 1987, Vol.46, No.4, 1819-1840.

[13] Grimble, M.J. , *Generalized Multivariable Control Algorithm for H_∞ Robust Design*, European Control Conference, Grenoble, France, 2-5 July 1991.

[14] Grimble, M.G., *LQG Optimal Control Design for Uncertain Systems*, IEE Part-D, Vol.139, No.1, January 1992.

[15] Grimble, M.J. and M.A. Johnson, H_∞ *Robust Control Design: A Tutorial Review*, Computing and Control Engineering Journal, November 1991.

[16] Kreyszig, E., *Advanced Engineering Mathematics*, John Wiley & Sons, Canda, 1993.

[17] Kucera, V. , *Discrete Linear Control*, John Wiley & Sons, 1979.

[18] Kwakernaak, H., *Robustness Optimization of Linear Feedback Systems*, Dept of Mathematics, Twente University, Memorandom NR-418, Jan 1983.

[19] Kwakernaak, H., *Minimax Frequency Domain Performance and Robustness Optimization of Linear Feedback Systems*, IEEE Trans. Aut, Control, Vol.Ac-30, No.10, October 1985.

[20] Maciejowski, J.M., *Multivariable Feedback Design*, Addison Wesley, 1989.

[21] Vidyasagar, M., *Control Systems Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA., 1985.

[22] Youla, D.C., J.J. Bongiorno and H.A. Jabr, *Modern Wiener-Hopf design of Optimal Controllers, Part I: The single-input single-output case*, IEEE Trans. Aut. Control, Ac-21, No.1, pp 3-13, February 1976.

[23] Zames, G., *Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses*, IEEE Trans. Aut. Cont., Vol. Ac-26, No.2, April 1981.