An Optimal Control of The Predation Model

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Abstract

This paper presents an extension to the predation model using the optimal control theory to obtain the optimal paths for the prey and predator levels for each type. These levels were represented as state variables for the optimal control problem, and also for the optimal paths for the levels of other prey and predator which were represented as control variables. In this direction, we will use the Maple program to solve the numerically controlled system based on nonlinear ordinary differential equations and using some constraints on the prey and predator numbers. The objective function is determined to reduce the lost numbers of all prey and predator at the end of the predation period to a minimum value in specific area. There are three different cases examined to reflect the effect of presence or absence the carrying capacity for the prey, and the effect of presence non-natural deterioration for the two species, and finally the effect of additional migration of two species.

1 Introduction:

The simple type of the predation model is describing the process of predation between the prey and the predator at the specified area. The model equations frequently are used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as a prey. This model makes a number of assumptions, not necessarily realizable in nature, about the environment and evolution of the predator and prey populations:

The prey finds ample food at all times. The food supply of the predator depends entirely on the size of the prey. The rate of change of population is proportional to its size. During the process, the environment does not change. The predators have limitless appetite.

Keywords: Predation model, Natural deterioration, Migration rate, Optimal control problem, Nonlinear ordinary differential equations, Minimization.

As nonlinear ordinary differential equations, the solution is deterministic and continuous. This implies that the generations of both predator and prey are continually overlapping.

The next equations represent the instantaneous growth rates of the prey and the predator:

$$\dot{x} = \frac{dx}{dt} = \alpha x - \beta \cdot x \cdot y$$

$$\dot{y} = \frac{dy}{dt} = \delta \cdot x \cdot y - \gamma \cdot y$$

Where $\{x , y\}$ represent the numbers of prey and predator respectively, and $\{\frac{dx}{dt}, \frac{dy}{dt}\}$ represent the instantaneous growth rates of prey and predator respectively, and $\{\alpha, \beta, \delta$ and $\gamma\}$ are real positive parameters. The prey are assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation. This exponential growth is represented above by the term $\{\alpha x\}$. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet. This is represented above by the term $\{\beta \cdot x \cdot y\}$. If either x or y is zero then there can be no predation process. With these two terms the equation above can be interpreted as follows: the rate of change of the prey's population is given by its own growth rate minus the rate at which it is preyed upon. In the second equation above, the term $\{\delta \cdot x \cdot y\}$ represents the growth of the predator population. Note the similarity to the predation rate. However, a different constant is used as the rate at which the predator population grows, is not necessarily equal to the rate at which it consumes the prey. The term $\{\gamma \cdot y\}$ represents the loss rate of the predators. It leads to an exponential decay in the absence of prey. Hence the equation expresses that the rate of change of the predator's population depends upon the rate at which it consumes prey minus its intrinsic death rate. In the model system, the predators thrive when there is plentiful the prey but, ultimately, outstrip their food supply and decline. As the predator population is low the prey population will increase again. These dynamics continue in a cycle of growth and decline. The predation model is referred as Lotka–Volterra model. The equations of this model is an example of Kolmogorov model [7, 5, 12] which is a more general framework that can model the dynamics of ecological systems with predator–prey interactions, competition, disease, and mutualism. The Lotka–Volterra model was initially proposed by Lotka in the theory of autocatalytic chemical reactions in 1910 [14, 10]. The model was later extended to include density dependent prey growth and a functional response of the form developed, a model that has become known as the Rosenzweig and McArthur model [17]. Both the Lotka–Volterra and Rosenzweig-MacArthur models have been used to explain the dynamics of natural populations of predator and prey [9, 13]. In the late 1980s, an alternative to the Lotka–Volterra predator-prey model is Arditi and Ginzburg model [2]. The Lotka–Volterra equations have a long history of use in economic theory [1,6,8,11].

Alexandra [3] proposed two extensions of the Lotka-Volterra competition model [15, 18]. The first one is inspired by the innovation component that is a fundamental part of the standard Bass model [4] and permits one of the principal drivers of adoptions in the markets to be taken into account.

The second extension allows that the Lotka-Volterra model may become diachronic by simply adjoining a standard Bass model [4] that is able to capture the diffusion of the first competitor in its stand-alone period.

We could cite, among others, a work by Morris and Pratt [16] that proposed an application of the Lotka-Volterra competition model in a market in which populations are competitors that contend for market shares to obtain a competitive advantage. Their analysis describes the evolution of the diffusion model of the second competitor that invades the market with respect to the first competitor as a function of parameters of the models, classifying the final situation in defined classes. In this case, they consider forcedly the two competitors as if they were synchronous, even if it is not clearly so.

The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, and other necessities available in the environment.

The carrying capacity was originally used to determine the number of animals that could graze on a segment of land without destroying it. Later, the idea was expanded to more complex populations, like humans. The carrying capacity of an environment may vary for different species and may change over time due to a variety of factors including: food availability, water supply, environmental conditions and living space.

This paper presents the predation model with some modifications using the optimal control theory. We presented an objective function which represents the lost numbers of all species during the predation period. Also, specifying the states variables which represent the levels of two origin species.

The control variables are represented as another prey and predator at the same specified area. We will use three cases for this model. The first one presents the model without carrying capacity for the prey, and, without non-natural deterioration for two species and additive migration for the two species. The second one presents the non-natural deterioration for the two species, additional to the presences of carrying capacity for the prey. Note that: the deterioration rates are non-natural deterioration rates but resulting in the predation process and any circumstances in the environment. Finally, the third one presents the model, additional to the presences of carrying capacity for the prey, with deterioration and migration.

Note that: the migration rates are additive migration that exceeds the numbers of two species.

This paper is organized as follow: Sections 2, 3 and 4 present the first, second and third cases respectively with a numerical examples for each case. Finally, Section 5 summarizes the results and gives some conclusions.

2. First Case

In this section, we will present the model without deterioration, migration and carrying capacity for the prey. We will use the variable x_1 to represent the numbers of prey and the variable x_2 to represent the numbers of predator. An objective function is minimizing the total lost numbers, $x_0(T)$, of all species at the end of the predation period T according to some constraints on the rates of growth of prey and predator.

We can use the following notations:

$x_1(t)$: Prey's numbers at the time *t*.

$x_2(t)$: Predator's numbers at the time t.

$u_1(t)$: Another prey's numbers at the time *t* .

$u_2(t)$: Another predator's numbers at the time *t*.

α : Growth rate of prey.

β : Ingestion rate of predators.

δ : Prey's and predator's assimilation efficiency.

γ : Predator's mortality rate.

q_1 : Presence rate of another prey.

q_2 : Presence rate of another predator.

t : Time.

T : Length of predation period.

$x_0(t)$: Lost numbers of all species at time *t*.

The total lost function, which we need to minimize, is determined as:

$$x_0(\mathrm{T}) = \int_o^T \left[\alpha x_1^2(t) - \gamma x_2^2(t) - \delta\beta x_1(t) x_2(t) + q_1 q_2 u_1(t) u_2(t)\right] dt. \quad (1)$$

Subject to:

$$\dot{x}_1 = \frac{dx_1}{dt} = x_1\left[\alpha - \beta x_2\right] + q_1 u_1. \qquad (2)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = x_2\left[\delta x_1 - \gamma\right] + q_2 u_2. \qquad (3)$$

For All parameters $\alpha, \beta, \delta, \gamma, q_1$ and q_2 are positive real values.

According to the law of Forest, the predator may kill the prey and not eat it, and may eat another prey, and may itself dies, for one reason or another, after killing it. For this reason there may be some or all parts of the prey or the predator endless. They end only by the end of the predation period. This may be resulted in the presence of the some negative values of both prey and predator. Because of the lost here is considered the death of the prey or the predator not eat them. But the body did not end yet and of course this reflects on the eating process for each one. So, it is not strange that occurrence some negative values in the state variables *X's* or the control variables *U*'s during the predation period.

As an optimal control problem, we will use the Pontryagin Principle to solve this problem. Let us define $\frac{dx_0(T)}{dt} = \dot{x}_0$, and introduce the co-state variables $\lambda_0(t)$, $\lambda_1(t)$ and $\lambda_2(t)$ corresponding to the state variables $x_0(t)$, $x_1(t)$ and $x_2(t)$ respectively. We can write the Hamiltonian function from equations (1), (2) and (3) as follows:

$$H = \lambda_0 \dot{x}_0 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 \qquad (4)$$

In addition, we obtained the co-state equations and the Lagrange function as follows:

$$L = H + \mu_1 x_1 + \mu_2 x_2 + \mu_3 u_1 + \mu_4 u_2 \qquad (5)$$

where, $\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)$ are known as Lagrange multipliers.

These Lagrange multipliers satisfy the conditions:

$$\mu_1 x_1 = 0, \quad \mu_2 x_2 = 0, \quad \mu_3 u_1 = 0, \quad \mu_4 u_2 = 0. \qquad (6)$$

We can easily obtain the co-state equations

$$\dot{\lambda}_i(t) = -\frac{\partial L}{\partial x_i}, \quad i = 0,1,2, \tag{7}$$

then,

$$\dot{\lambda}_0 = -\frac{\partial L}{\partial x_0} = 0, \quad \dot{\lambda}_1 = -\frac{\partial L}{\partial x_1}, \quad \dot{\lambda}_2 = -\frac{\partial L}{\partial x_2}, \tag{8}$$

The first equation of the system (8) shows that the co-state variable $\lambda_0(t)$ remains constant along the optimal trajectory, and the Pontryagin principle requires that this constant should be a negative value (Minimization problem) as:

$$\lambda_0(t) = -1. \tag{9}$$

Since $x_1(t), x_2(t), u_1(t), u_2(t) \neq 0$, then from the constraint (6), we have

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0. \tag{10}$$

We can write the Lagrange function L from the equations (1),(2),(3),(4),(5),(9) and (10) as following :

$$\begin{aligned} L = & -\left(\alpha x_1^2 - \gamma x_2^2 - \delta\beta x_1 x_2 + q_1 q_2 u_1 u_2\right) \\ & + \lambda_1\left(x_1\left[\alpha - \beta x_2\right] + q_1 u_1\right) \\ & + \lambda_2\left(x_2\left[\delta x_1 - \gamma\right] + q_2 u_2\right) \end{aligned} \tag{11}$$

From the equations (8) and (11), we have :

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial x_1} = \lambda_1\left(\beta x_2 - \alpha\right) - \lambda_2 \delta x_2 + 2\alpha x_1 - \delta\beta x_2, \tag{12}$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial x_2} = \lambda_1 \beta x_1 + \lambda_2\left(\gamma - \delta x_1\right) - 2\gamma x_2 - \delta\beta x_1, \tag{13}$$

with terminal conditions: $\lambda_i(T) \neq 0, \quad i = 1,2$.

To obtain the optimal levels of control variables $u_i(t), \ i = 1,2$, we differentiate the Lagrange function (11) with respect to u_1, u_2 respectively and put it equal to zero, we get

$$\frac{\partial L}{\partial u_1} = -q_1 q_2 u_2 + \lambda_1 q_1 = 0,$$
$$\frac{\partial L}{\partial u_2} = -q_1 q_2 u_1 + \lambda_2 q_2 = 0.$$

Then, the obtimal conctrol variables are become

$$u_1^*(t) = \frac{\lambda_2(t)}{q_1}, \qquad u_2^*(t) = \frac{\lambda_1(t)}{q_2}, \qquad q_1, q_2 \neq 0. \tag{14}$$

From equations (2),(3),(12),(13) and (14) we have the controlled system of non-linear ordinary differential equations as:

$$\left.\begin{aligned}
\dot{x}_1 &= x_1(\alpha - \beta x_2) + \lambda_2 \\
\dot{x}_2 &= x_2(\delta x_1 - \gamma) + \lambda_1 \\
\dot{\lambda}_1 &= \lambda_1(\beta x_2 - \alpha) - \lambda_2 \delta x_2 + 2\alpha x_1 - \delta\beta x_2 \\
\dot{\lambda}_2 &= \lambda_1 \beta x_1 + \lambda_2(\gamma - \delta x_1) - 2\gamma x_2 - \delta\beta x_1
\end{aligned}\right\}. \tag{15}$$

The total lost function can be determined using (1) and (14) as:

$$x_0(T) = \int_o^T \left[\alpha x_1^2(t) \;\; - \gamma x_2^2(t) \;\; - \delta\beta x_1(t) x_2(t) + \lambda_1(t)\lambda_2(t)\right] dt.$$

The system (15) can be used to describe the time evolution of numbers of the species in the first case.

This system can be solved numerically using Runge-Kutta method with initial values and terminal conditions:

$x_1(0)$	$x_2(0)$	$x_0(0)$	$\lambda_1(T)$	$\lambda_2(T)$
1	1	0	10	10

2.1 Numerical Example

We solve the system (15) numerically depending parameter's values:

α	β	δ	γ	q_1	q_2	T
0.9	0.1	0.1	0.5	0.1	0.1	10

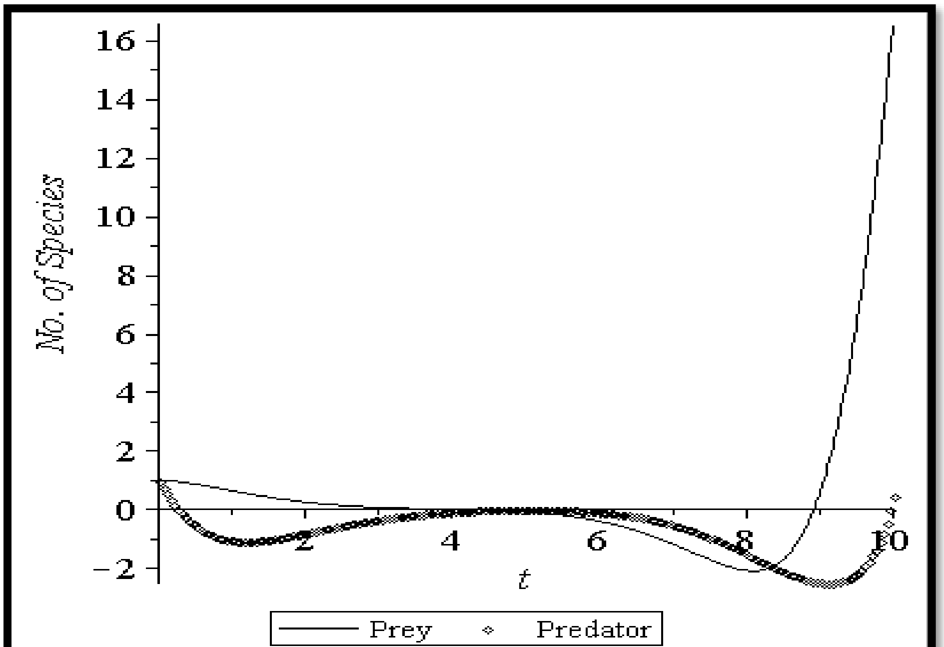

Fig.2.1 Species against time t

Figure 2.1 shows the behavior of Prey and Predator numbers during the predation period in the first case (absence of the deterioration and migration without carrying capacity for the prey).

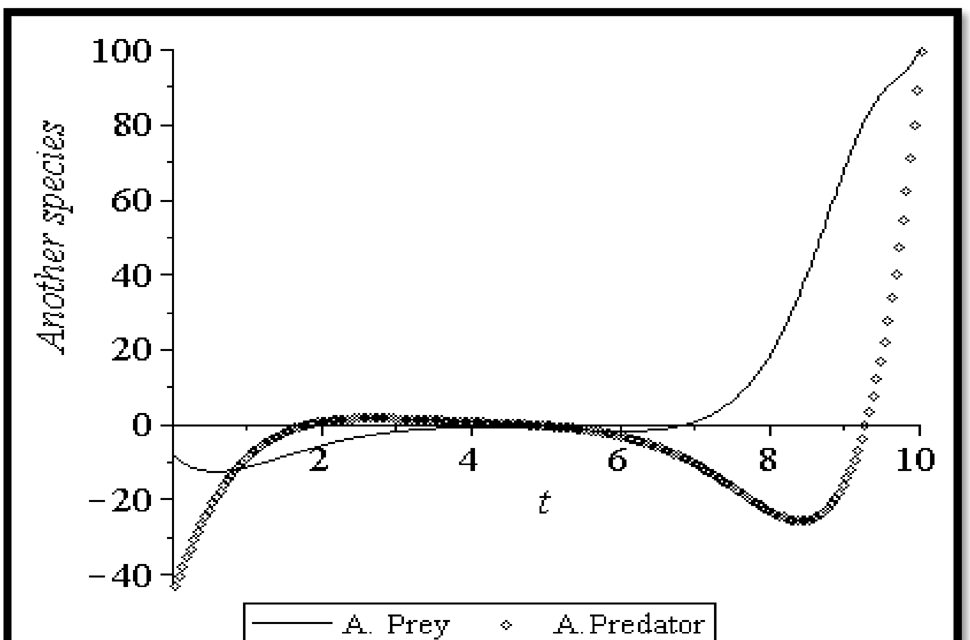

Fig.2.2 Another species against time t

Figure 2.2 shows the behavior of A. Prey and A. Predator numbers during the predation period in the first case (absence of the deterioration and migration without carrying capacity for the prey).

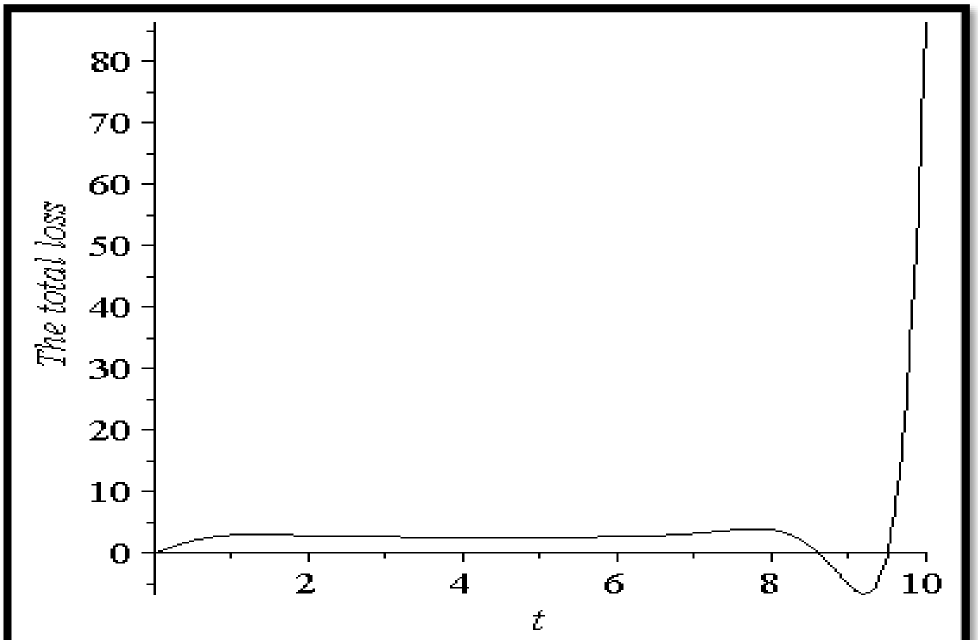

Figure 2.3 The total loss against time t

Figure 2.3 shows the behavior of total lost numbers function during the predation period in the first case (absence of the deterioration and migration without carrying capacity).

3. Second Case

In this case we will use the deterioration rates for the two species and the carrying capacity for the prey. In this section, we add the deterioration rates (θ_1, θ_2) for Prey and Predator respectively:

Min
$$J = x_0 = \int_o^T \Big[(\alpha - \theta_1) x_1^2(t) - (\gamma + \theta_2) x_2^2(t) - \delta\beta x_1(t) x_2(t) + q_1 q_2 u_1(t) u_2(t) \Big]\, dt. \tag{16}$$

Subject to:

$$\dot{x}_1 = \frac{dx_1}{dt} = x_1 \Big[(\alpha - \theta_1)(1 - \frac{x_1}{k}) - \beta x_2 \Big] + q_1 u_1. \tag{17}$$

$$\dot{x}_2 = \frac{dx_2}{dt} = x_2 \big[\delta x_1 - \gamma - \theta_2 \big] + q_2 u_2. \tag{18}$$

For All parameters $\alpha, \beta, \delta, \gamma, q_1, q_2, \theta_1$ and θ_2 are positive real values.

Since:

θ_1: Non-natural deterioration of the prey.

θ_2: Non-natural deterioration of the predator.

k : Carrying capacity of the prey.

In this case, the population of prey progressively increases to the limit k when $t \to \infty$.
Also, we can write L using the equations (16), (17) and (18):

$$L = -\left[(\alpha-\theta_1)x_1^{\,2} - (\gamma+\theta_2)x_2^{\,2} - \delta\beta x_1 x_2 + q_1 q_2 u_1 u_2\right]$$
$$+\lambda_1\left(x_1\left[(\alpha-\theta_1)(1-\frac{x_1}{k}) - \beta x_2\right] + q_1 u_1\right) \tag{19}$$
$$+\lambda_2\left(x_2\left[\delta x_1 - \gamma - \theta_2\right] + q_2 u_2\right).$$

From the equations (8) and (11), we have :

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial x_1} = \lambda_1\left(\beta x_2 + \frac{2(\alpha-\theta_1)}{k}x_1 - \alpha + \theta_1\right) - \lambda_2 \delta x_2 \quad , \tag{20}$$
$$+2(\alpha-\theta_1)x_1 - \delta\beta x_2$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial x_2} = \lambda_1 \beta x_1 + \lambda_2\left(\gamma + \theta_2 - \delta x_1\right) \quad , \tag{21}$$
$$-2(\gamma+\theta_2)x_2 - \delta\beta x_1$$

with terminal conditions $\lambda_i(T) \neq 0, \quad i = 1,2$.

The optimal levels of control variables $u_i(t), \quad i = 1,2$, as in the equation (14).

The controlled system of non-linear ordinary differential equations is become:

$$\left.\begin{aligned}
\dot{x}_1 &= x_1\left[(\alpha-\theta_1)(1-\frac{x_1}{k})-\beta x_2\right]+\lambda_2 \\
\dot{x}_2 &= x_2\left[\delta x_1-\gamma-\theta_2\right]+\lambda_1 \\
\dot{\lambda}_1 &= \lambda_1\Big(\beta x_2+\frac{2(\alpha-\theta_1)}{k}x_1-\alpha+\theta_1\Big)-\lambda_2\delta x_2 \\
&\quad +2(\alpha-\theta_1)x_1-\delta\beta x_2 \\
\dot{\lambda}_2 &= \lambda_1\beta x_1+\lambda_2\big(\gamma+\theta_2-\delta x_1\big) \\
&\quad -2(\gamma+\theta_2)x_2-\delta\beta x_1
\end{aligned}\right\}. \tag{22}$$

The total lost function using (14) and (16) has become:

$$x_0=\int_o^T\left[(\alpha-\theta_1)x_1^{\,2}(t)-(\gamma+\theta_2)x_2^{\,2}(t)-\delta\beta x_1(t)x_2(t)\right.$$
$$\left.+\lambda_1(t)\lambda_2(t)\right]dt.$$

The system (22) can be used to describe the time evolution of levels of species in the second case. This system can be solved numerically using Runge-Kutta method with initial values and terminal conditions:

$x_1(0)$	$x_2(0)$	$x_0(0)$	$\lambda_1(T)$	$\lambda_2(T)$
1	1	0	10	10

3.1 Numerical Example

We will solve the system (22) numerically using the parameter's values:

α	β	δ	γ	q_1	q_2	θ_1	θ_2	T	k
0.9	0.2	0.2	0.5	0.1	0.1	0.01	0.01	10	10

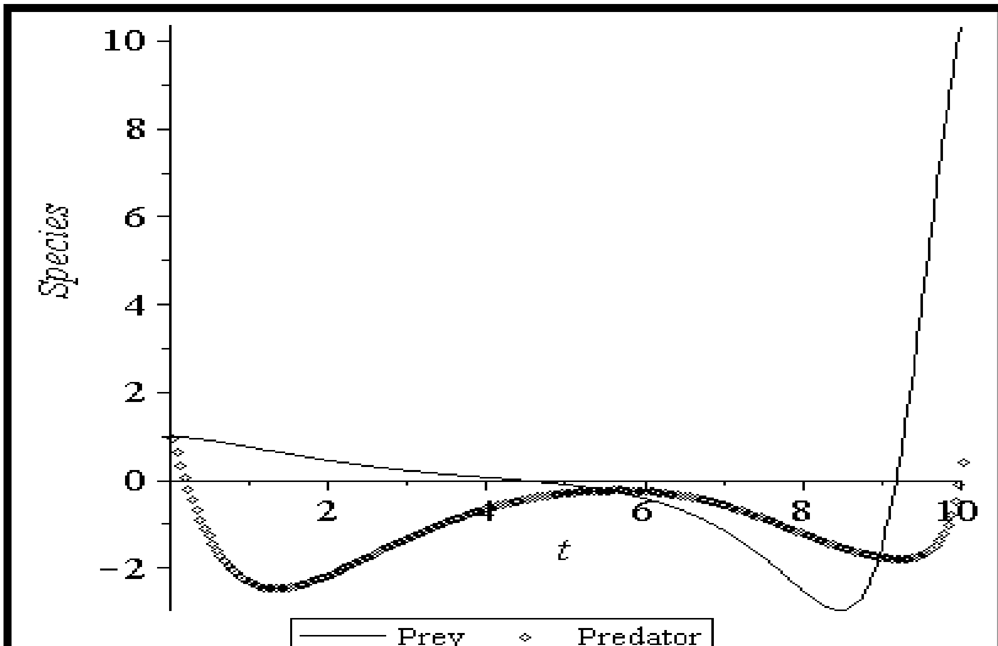

Fig.3.1 Species against time t

From Figure 3.1, shows the behavior of Prey and Predator numbers during the predation period in the second case (exist of the deterioration with carrying capacity for the prey).

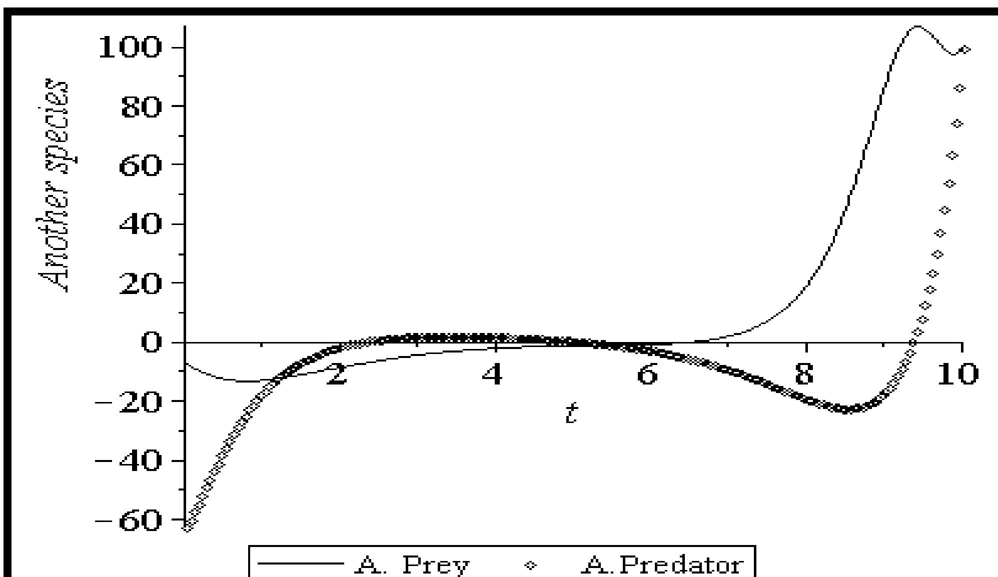

Fig.3.2 Another species against time t

From Figure 3.2, shows the behavior of A. Prey and A. Predator numbers during the predation period in the second (exist of the deterioration with carrying capacity for the prey).

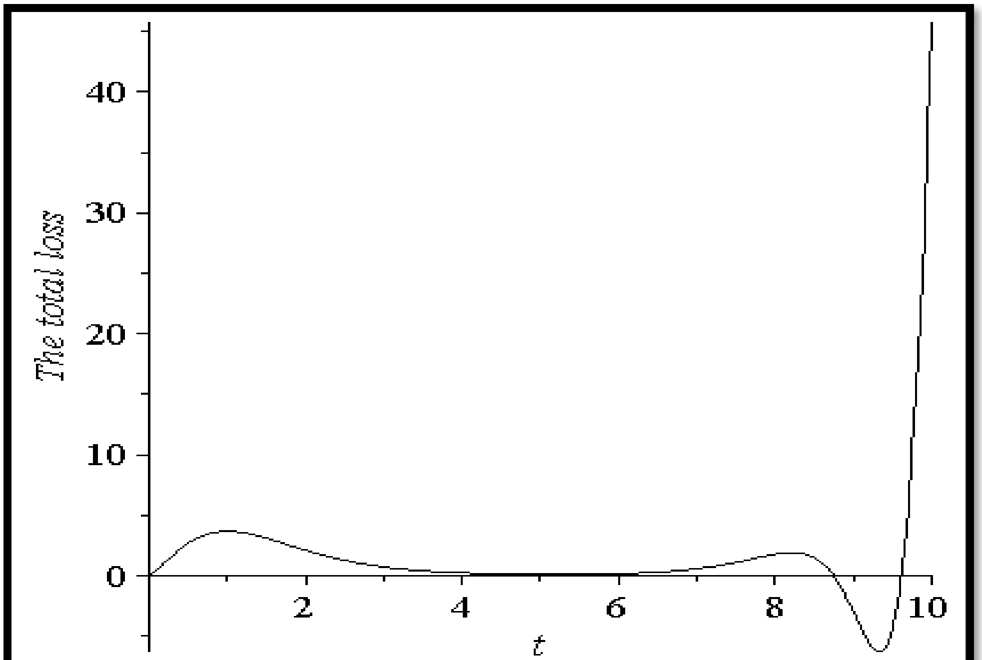

Fig.3.3 The total loss against time t

From Figure 3.3, shows the behavior of total lost numbers function during the predation period in the second case (exist of the deterioration with carrying capacity).

4. Third Case

In this case we will use the deterioration rates for the two species and the additive migration rates of the two species, and the carrying capacity for the prey:

In this section, beside to the deterioration rates (θ_1, θ_2), we add the rates of additive migration (φ_1, φ_2) for prey and the predator respectively:

Min
$$J = x_0 = \int_o^T \Big[(\alpha - \theta_1 + \varphi_1) x_1^2(t) - (\gamma + \theta_2 - \varphi_2) x_2^2(t) - \delta\beta x_1(t) x_2(t) + q_1 q_2 u_1(t) u_2(t) \Big] \, dt. \tag{23}$$

Subject to:

$$\dot{x}_1 = \frac{dx_1}{dt} = x_1 \Big[(\alpha - \theta_1 + \varphi_1)(1 - \frac{x_1}{k}) - \beta x_2 \Big] + q_1 u_1. \tag{24}$$

$$\dot{x}_2 = \frac{dx_2}{dt} = x_2 \Big[\delta x_1 - (\gamma + \theta_2 - \varphi_2) \Big] + q_2 u_2. \tag{25}$$

For All parameters $\alpha, \beta, \delta, \gamma, q_1, q_2, \theta_1$ and θ_2 are positive real values.

Since:

φ_1: Migration rate of prey.

φ_2: Migration rate of predator.

Also, we can write L using the equations (23), (24) and (25)

$$L = -\left[(\alpha - \theta_1 + \varphi_1)x_1^2 - (\gamma + \theta_2 - \varphi_2)x_2^2 - \delta\beta x_1 x_2 + q_1 q_2 u_1 u_2\right] + \lambda_1\left(x_1\left[(\alpha - \theta_1 + \varphi_1)(1 - \frac{x_1}{k}) - \beta x_2\right] + q_1 u_1\right) + \lambda_2\left(x_2\left[\delta x_1 - (\gamma + \theta_2 - \varphi_2)\right] + q_2 u_2\right). \tag{26}$$

From the equations (8) and (19), we have :

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial x_1} = \lambda_1\left(\beta x_2 + \frac{2(\alpha - \theta_1 + \varphi_1)}{k}x_1 - \alpha + \theta_1 - \varphi_1\right) - \lambda_2 \delta x_2 + 2(\alpha - \theta_1 + \varphi_1)x_1 - \delta\beta x_2, \tag{27}$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial x_2} = \lambda_1 \beta x_1 + \lambda_2\left(\gamma + \theta_2 - \varphi_2 - \delta x_1\right) - 2(\gamma + \theta_2 - \varphi_2)x_2 - \delta\beta x_1, \tag{28}$$

with trerminal conditions $\lambda_i(T) \neq 0, \quad i = 1,2$.

The optimal levels of control variables $u_i(t), \quad i = 1,2$, as in the equation (14).

The controlled system of non-linear ordinary differential equations is become:

$$
\left.\begin{aligned}
\dot{x}_1 &= x_1\left[(\alpha-\theta_1+\varphi_1)(1-\frac{x_1}{k})-\beta x_2\right]+\lambda_2 \\
\dot{x}_2 &= x_2\left[\delta x_1-(\gamma+\theta_2-\varphi_2)\right]+\lambda_1 \\
\dot{\lambda}_1 &= \lambda_1\left(\beta x_2+\frac{2(\alpha-\theta_1+\varphi_1)}{k}x_1-\alpha+\theta_1-\varphi_1\right) \\
&\quad -\lambda_2\delta x_2+2(\alpha-\theta_1+\varphi_1)x_1-\delta\beta x_2 \\
\dot{\lambda}_2 &= \lambda_1\beta x_1+\lambda_2\left(\gamma+\theta_2-\varphi_2-\delta x_1\right) \\
&\quad -2(\gamma+\theta_2-\varphi_2)x_2-\delta\beta x_1
\end{aligned}\right\}, \qquad (29)
$$

The total lost function has become using (14) and (23):

$$
x_0=\int_o^T\left[(\alpha-\theta_1+\varphi_1)x_1^2(t)-(\gamma+\theta_2-\varphi_2)x_2^2(t)\right.
$$
$$
\left.-\delta\beta x_1(t)x_2(t)+\lambda_1(t)\lambda_2(\mathrm{t})\right]dt.
$$

The system (29) can be used to describe the time evolution of levels of species in the third case. This system can be solved numerically using Runge-Kutta method with initial values and terminal conditions:

$x_1(0)$	$x_2(0)$	$x_0(0)$	$\lambda_1(T)$	$\lambda_2(T)$
1	1	0	10	10

4.1 Numerical Example

We will solve the system (29) numerically using the parameter's values:

α	β	δ	γ	q_1	q_2
0.9	0.3	0.3	0.5	0.1	0.1
θ_1	θ_2	φ_1	φ_2	T	k
0.01	0.01	0.1	0.1	10	10

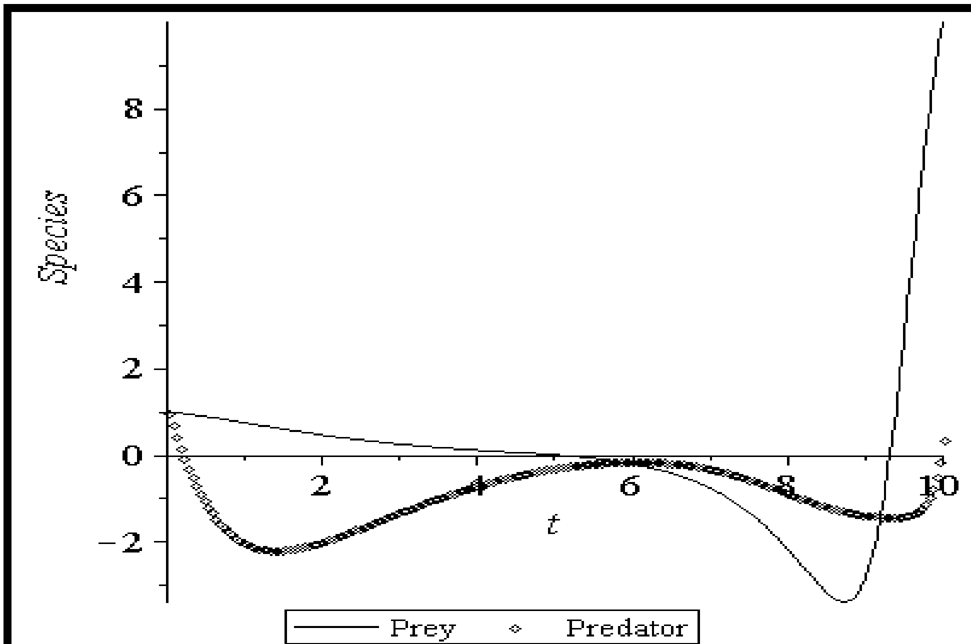

Fig. 4.1 Species against time t

From Figure 4.1, shows the behavior of prey and predator numbers during the predation period in the third case (exist of the deterioration and migration with carrying capacity for the prey).

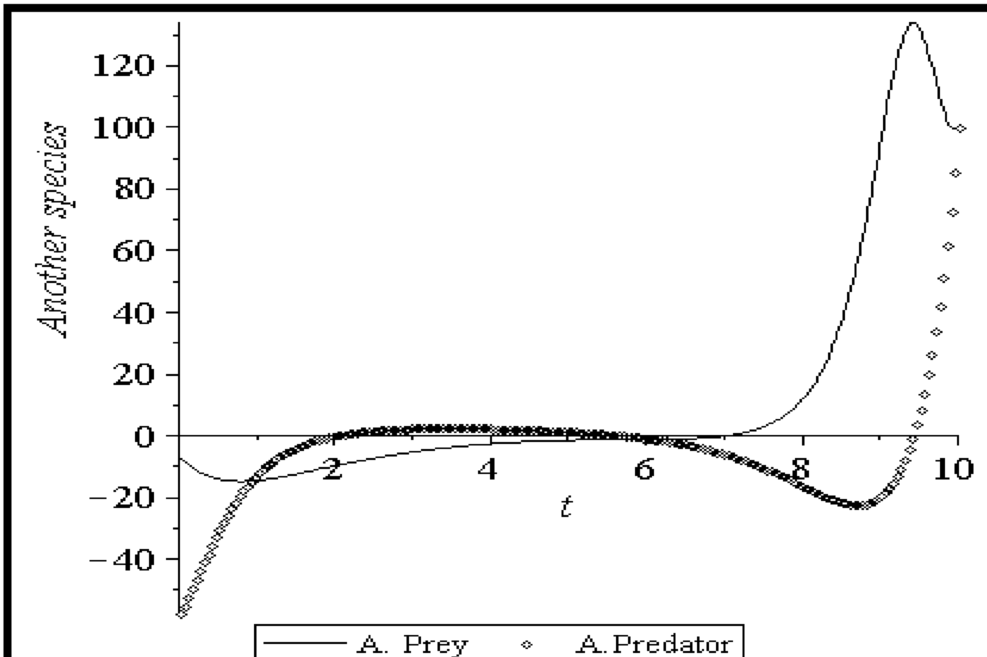

Fig. 4.2 Another species against time t

From Figure 4.2, shows the behavior of A. Prey and A. Predator numbers during the predation period in the third case (exist of the deterioration and migration with carrying capacity for the prey).

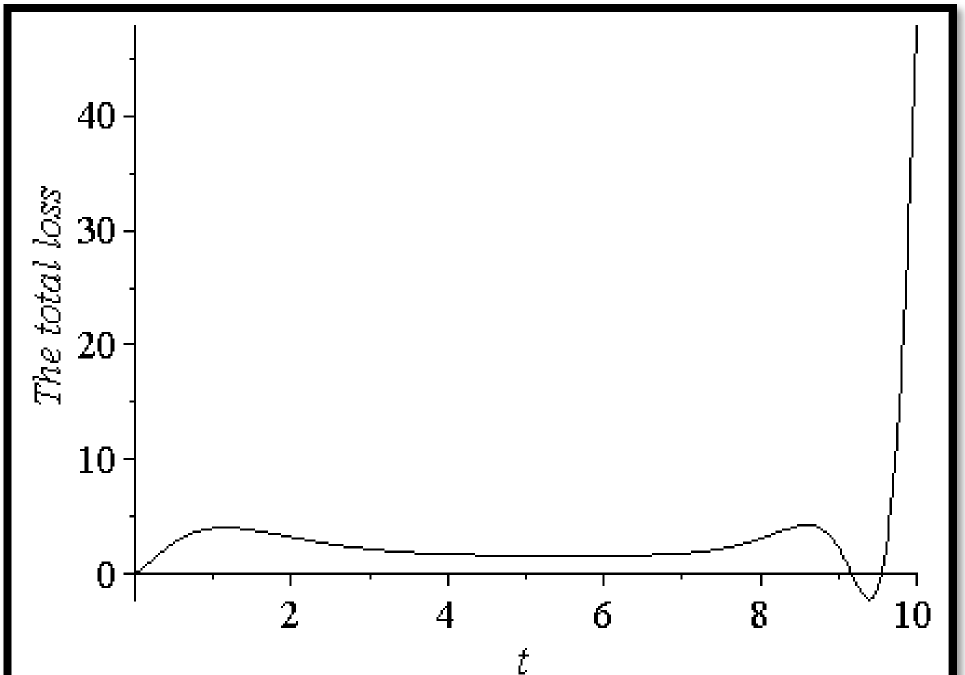

Fig.4.3 The total loss against time t

From Figure 4.3, shows the behavior of total lost numbers function during the predation period in the third case (exist of the deterioration and migration with carrying capacity for the prey).

5. Conclusions

From the numerical examples we can summarize all results for three cases about an optimal solution in the next table:

Optimal Values	First Case	Second Case	Third Case
X_1^*	16	10	10
X_2^*	0.49	0.43	0.36
U_1^*	100	100	100
U_2^*	100	100	100
x_0^*	86	46	48

We can conclude that:

1)- The numbers of prey are almost equal in the second and third cases but increasing in the first case. Indicating that they are affected by the migration and the deterioration.

2)- The numbers of predator decreases slightly in all three cases, indicating that the number of predators is relatively un-affected by the migration and the deterioration.

3)- The numbers of A. prey and A. predator did not affect by the deterioration and migration and remained constant.

4)- The lost numbers of all species have been affected by the rates of deterioration and migration. The maximum lost is happened in the first case when the deterioration and migration are absence, without carrying capacity. But the minimum lost has reached in the second case when the migration does not exist.

5)- The numerical solution can only be achieved if the ingestion rate β and assimilation efficiency δ are increased from 0.1 in the first case, to 0.2 in the second case, to 0.3 in the third case. This is justified because of existing the migration and the deterioration.

6)- Finally, we noted that the carrying capacity of prey has affected on their sizes at the end of predation period. As we noticed in the second and the third cases that the levels of prey close to 10 in the case of presence the carrying capacity for the prey. But because of absence the carrying capacity, in the first case, we have found that the levels close to 16.

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