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Bit Error Probability of an Interleaved Convolutional Coded BPSK Signal Over a Correlated Fading Channel

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ABSTRACT

This paper analyzes the performance of an interleaved convolutional coded BPSK with perfect channel state information and coherent detection, assuming perfect synchronization. Analytical results for the pairwise error event probability using the characteristic function technique is derived. An approximate bit error probability is obtained by summing the probabilities of the dominant error events.

I. Introduction

Most of the well known codes, including the convolutional codes, are effective when the errors caused by the channel are statistically independent [1]. However there are many channels that exhibit bursty error characteristics [2]. Signal fading due to time variant multipath propagation often causes the signal to fall below the noise level, thus resulting in a large number of errors.

An effective method for dealing with burst error channels is to interleave the coded bits in such a way that the bursty channel is to be transformed into a channel having independent errors. In practical systems, full interleaving cannot be achieved completely and a residual amount of correlation remains affecting the system performance.

In this paper, we investigate the probability of error of an interleaved convolutional coded BPSK over a correlated multipath fading channel with perfect channel state information (CSI) and coherent detection.

The rest of the paper is organized as follow. In sec. II, the system model is introduced and the decoding error is derived, In sec. III, the probability density function of the decoding error variable is derived using the characteristic function technique. In sec. IV, the pairwise error event probability is obtained. The bit error probability is given in sec. V. Conclusions are drawn in sec. VI.

II. System Model

The block diagram of the used system is shown in the Fig.1.

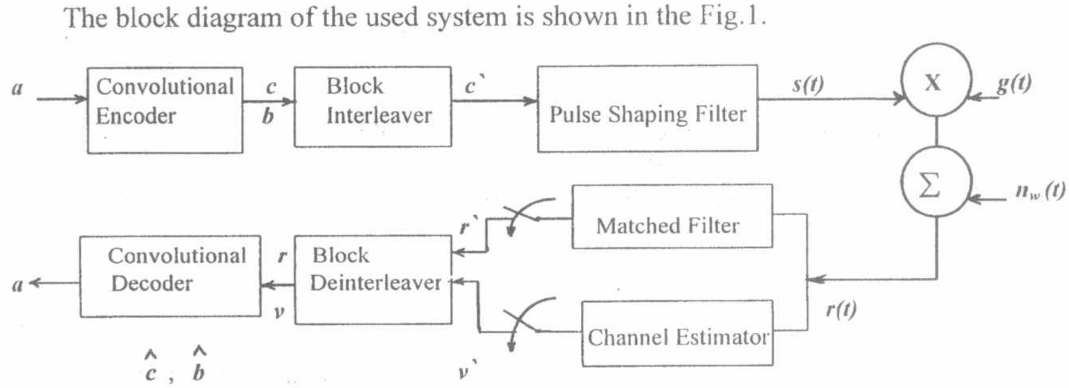

Fig. 1 System block diagram of an interleaved convolutional coded BPSK signal over a correlated fading channel.

The input to the linear convolutional encoder (**n, l,** ν) is a sequence of binary digits given by **a** = ($a_1, a_2, \ldots, a_k, \ldots$) and its corresponding output is

$$b = (b_1^{\ 1}, b_1^{\ 2}, \ldots, b_1^{\ n}\ b_2^{\ 1}, b_2^{\ 2}, \ldots, b_2^{\ n}, \ldots b_k^{\ 1}, b_k^{\ 2}, \ldots, b_k^{\ n}, \ldots\ldots) \tag{1}$$

To simplify the analysis we use the transformation

$$b_j^{\ i} = c_k \quad , \quad k = (j-1) * n + i \tag{2}$$

and therefore, the encoder output can be represented by a binary codeword c of length **M** i.e.,

$$c = (\ c_1, c_2, \ldots, c_k, \ldots, \ c_M) \tag{3}$$

In order to disperse possible deep fades in the channel, the codeword c is passed to a block interleaver with a buffer of size α rows and β columns where β is the interleaving depth, and $\alpha \times \beta = M$. The binary symbols in the codeword c will fill the interleaver buffer column by column, while the output sequence of the interleaver is obtained by reading out the contents of the buffer row by row. The interleaver output sequence is

$$c` = (c_1`, c_2`, \ldots, c_k`, \ldots, c_M`) \tag{4}$$

The c_k`' s are fed to a pulse shaping filter at a rate of ***1/T*** symbol per second. The impulse response of the filter is ***A p(t)*** where ***A*** is a constant representing the signal amplitude and ***p(t)*** is a unit energy pulse. We assume that the concatenation of ***p(t)*** and the receiver matched filter is a pulse that satisfies the Nyquist's criterion for zero intersymbol interference. The transmitted signal is

$$s(t) = A \sum_{k=1}^{M} c_k`\ p(t - kT) \tag{5}$$

and the corresponding received signal is

$$r(t) = g(t)\ s(t) + n_w(t) \tag{6}$$

where ***g(t)*** is a zero mean, complex Gaussian random process and $n_w(t)$ is the complex envelope of the channel's AWGN, with double-sided power spectral density of $N_o/2$. The received signal $r(t)$ is passed to a matched filter with an impulse response $p^*(t)/\sqrt{N_o}$. The output of the matched filter is sampled at the symbol rate ***1/T*** to produce the sequence $r` = (r_1`, r_2`, \ldots, r_k`, \ldots, r_M`)$. Assuming that the fading process is slow enough that it is approximately constant over each symbol interval, the received signal can be written as

$$r(t) = A \sum_{k=1}^{M} g_k` c_k`\ p(t - kT) + n_w(t) \tag{7}$$

where g_k` is the value of **g(t)** in the $k^{th}$ symbol interval. Consequently the sample $r_k$` can be written as

$$r_k` = u_k`\ c_k` + n_k` \tag{8}$$

where u_k`' s and $n_k$`'s are zero mean complex Gaussian random variables representing, respectively, the fading and the additive noise experienced by the k^{th} transmitted symbol. The n_k`'s are independent and identically distributed (iid) random variables with a unit variance. On the other hand, the $u_k$`'s are not independent. The autocorrelation function of the u_k`'s is

$$\rho(k) = \frac{E_s}{N_o} R(kT) \tag{9}$$

where ***R(kT)*** is the autocorrelation function of the fading process. For a mobile radio channel, the autocorrelation function ***R(kT)*** can be modeled as [3]

$$R(kT) = \exp(-|k f_D T|) \tag{10}$$

where f_D is the maximum Doppler frequency. The term (E_s/No) is the average signal-to-noise ratio .The samples in the sequence $r` = (r_1`, r_2`, ..., r_k`, ..., r_M`)$ are deinterleaved to produce the sequence

$r = (r_1, r_2, ..., r_k, ..., r_M)$ where

$$r_k = u_k c_k + n_k \tag{11}$$

The u_k*'s* in eq.(11) are a set of correlated zero mean Gaussian random variables. The channel estimator as shown in Fig. 1 will extract information about the channel complex gains from the received signal. Let the sequence that appears at the output of the estimator be denoted by $v` = (v_1`, v_2`, ... , v_k`, ..., v_M`)$ . If we assume an ideal situation, the estimator provides perfect channel state information (CSI). The sample in the sequence $v` = (v_1`, v_2`, ..., v_k`, ..., v_M`)$ are deinterleaved to produced the sequence $v = (v_1, v_2, ..., v_k, ..., v_M)$. Using the transformation given by eq.(2), the two sequences

$r = (r_1, r_2, ..., r_k, ..., r_M)$ and $v = (v_1, v_2, ..., v_k, ..., v_M)$ are transformed back into the forms

$$r = (r_1^{\ 1}, r_1^{\ 2}, ..., r_1^{\ n} r_2^{\ 1}, r_2^{\ 2}, ..., r_2^{\ n}, ... r_k^{\ 1}, r_k^{\ 2}, ..., r_k^{\ n}, ... r_M^{\ 1}, r_M^{\ 2}, ..., r_M^{\ n}) \tag{12}$$

and

$$v = (v_1^{\ 1}, v_1^{\ 2}, ..., v_1^{\ n} r_2^{\ 1}, v_2^{\ 2}, ..., v_2^{\ n}, ... v_k^{\ 1}, v_k^{\ 2}, ..., v_k^{\ n}, ... v_M^{\ 1}, v_M^{\ 2}, ..., v_M^{\ n}) \tag{13}$$

The two sequences r and v are fed to the input of the Viterbi decoder[4]. The Viterbi decoder will select the codeword c given by

$$\hat{c} = (\hat{c}_1, \hat{c}_2, ..., \hat{c}_k, ..., \hat{c}_M) \tag{14}$$

and correspondingly

$$\hat{b} = (\hat{b}_1^{\ 1}, \hat{b}_1^{\ 2}, ..., \hat{b}_1^{\ n} \hat{b}_2^{\ 1}, \hat{b}_2^{\ 2}, ..., \hat{b}_2^{\ n}, ... \hat{b}_k^{\ 1}, \hat{b}_k^{\ 2}, ..., \hat{b}_k^{\ n}, ... \hat{b}_M^{\ 1}, \hat{b}_M^{\ 2}, ..., \hat{b}_M^{\ n}) \tag{15}$$

whose metric

$$M_e(\hat{b}) = \sum_{j=1}^{M} \sum_{i=1}^{n} |r_j^{\ i} - v_j^{\ i} \hat{b}_j^{\ i}|^2 \tag{16}$$

is smallest. This decoding metric is optimum for coherent PSK with perfect channel state information [5]. If any decoded codeword c is not equal to the transmitted codeword c, the random variable

$$\begin{aligned} D &= M_e(\hat{b}) - M_e(b) \\ &= \sum_{j=1}^{M} \sum_{i=1}^{n} \{ r_j^{\ i} v_j^{\ i*} (b_j^{\ i*} - \hat{b}_j^{\ i*}) + r_j^{\ i*} v_j^{\ i} (b_j^{\ i} - \hat{b}_j^{\ i}) \} \end{aligned} \tag{17}$$

is less than zero, then a decoding error will occur. The probability that $D \leq 0$ is known as the pairwise error event probability.

III. The Probability Density Function of the Error Variable D

The characteristic function technique is used to find the probability density function of the error variable D and also the pairwise error event probability as follows.

Let $(k_1, k_2, \ldots, k_k, \ldots, k_L)$ be the set of index k for which the binary coded bits $c_k \neq \hat{c}_k$. L is referred to as the length of an error event while $S = k_L - k_1 + 1$ is referred to as the span of an error event. For coded binary ***PSK*** with perfect CSI, the random variable D in eq.(17) written using the index k is

$$D = \sum_{l=1}^{L} r_{kl} u_{kl}^{*} (c_{kl}^{*} - \hat{c}_{kl}^{*}) + r_{kl}^{*} u_{kl} (c_{kl} - \hat{c}_{kl}) \tag{18}$$

Substituting the value of r_k given by eq. (11) we get

$$D = \sum_{l=1}^{L} |z_l|^2 + z_l n_{kl}^{*} + z_l^{*} n_{kl} \tag{19}$$

where

$$z_l = u_{kl} d_l \quad , \quad d_l = c_{kl} - \hat{c}_{kl} \tag{20}$$

n_{kl}, $l = 1,2,\ldots,L$ are iid complex random variables each having a zero mean and a unit variance. Eq. (18) can be written in a matrix form as

$$D = \underline{z}^h \underline{z} + \underline{n}^h \underline{z} + \underline{z}^h \underline{n} \tag{21}$$

where $\underline{\mathbf{z}}$ and $\underline{\mathbf{n}}$ are column vectors whose components are the z_l's and n_{kl}'s and $\underline{\mathbf{z}}^h$ and $\underline{\mathbf{n}}^h$ are the Hermitian transposes of $\underline{\mathbf{z}}$ and $\underline{\mathbf{n}}$. The covariance matrix for the z_l's is

$$\Phi_{zz} = \tfrac{1}{2} E[\underline{z}\,\underline{z}^h] = \Delta\, \Phi\, \Delta^h \tag{22}$$

where $\mathbf{E}[\;]$ is the expected value and Δ is a diagonal matrix whose l^{th} component is equal to d_l and the matrix Φ is the covariance matrix for the u_{kl}'s.

For a linear convolutional code, the set of Hamming distances of the code sequences generated from the all zero code sequence is the same as the set of distances of the code sequences with respect to any other code sequence. Consequently, we assume without loss of generality that the all-zero code sequence is the transmitted one. Therefore the matrix Δ in eq. (22) will be an identity matrix with dimensional L.

As long as the number of rows α in the interleaver buffer is large, a dominant error event can seldom span two adjacent columns of the buffer. Therefore, if c_{ki} and c_{kj} are two coded bits among the L bits constituting the error event then the complex gains u_{ki} and u_{kj} experienced by these two symbols for mobile channel have a correlation equal

$$\Phi(k_i, k_j) = \frac{E_s}{N_o} \exp(-|(k_i - k_j)\,\beta f_n T|) \tag{23}$$

From eq. (23) we can say that the effect of interleaving is to increase the Doppler frequency by a factor of β. Regarding the noise component it can be shown that the covariance matrix for n_k's is an identity matrix of dimension L.

Since D is a sum of independent quadratic forms of complex Gaussian variety, then with a slight modification to eq. (B-3-16) in [6], the characteristic function of D can be shown equal to

$$\Phi_D(s) = \prod_{l=1}^{L} \left(\frac{-1}{\lambda_l}\right) \frac{1}{(s - p_{1l})(s - p_{2l})} \tag{24}$$

where λ_l, $l = 1,2,\ldots,L$ are the set of eigenvalues of Φ_{zz} and

$$\begin{bmatrix} p_{1l} \\ p_{2l} \end{bmatrix} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{\lambda_l}} \quad , \qquad l = 1,2,\ldots\ldots, L \tag{25}$$

are the poles of the characteristic function. It should be pointed out that when $\lambda_l = \mathbf{0}$, the term $\{ - \lambda_l (s - P_{1l}) (s - P_{2l}) \}^{-1}$ is equale to unity. The probabililty density function of D, $P_D(d)$, is simply the inverse Laplace transform of $\Phi_D(s)$, where $\Phi_D(s)$ is analytic through the finite s-plane except for a finite number of poles.

If L_R denotes a vertical line segment $s = r_o + j\,d \;\; (-R \le d \le R)$, where the constant r_o is positive and large enough so that the segment lies to the right of all of those poles, then the inverse Laplace transform $P_D(d)$ of $\Phi_D(s)$, defined for the real values of d is given by

$$P_D(d) = \frac{1}{2j\pi} \lim_{R \to \infty} \int_{L_R} \exp(sd)\, \Phi_D(s)\; ds \tag{26}$$

The choice of the positive number r_o is immaterial as long as L_R lies to the right of the poles of Φ_D. Since the function Φ_D is specified, equations (24) and (25), residues theorem [7] can be used to evaluate the limit in eq. (26). With some straight forward analytical manipulation we get

$$\lim_{R \to \infty} \int_{L_R} \exp(sd)\, \Phi_D(s)\; ds = 2j\pi \sum_{l=1}^{L} \operatorname{Res} [\, \exp(sd)\;\; \Phi_D(s)] \big|_{p_{l1}, p_{l2}} \tag{27}$$

Substituting the value of $\Phi_D(s)$ given by eq. (24) and then comparing with eq. (26), we get

$$P_D(d) = \begin{cases} \prod_{k=1}^{L} \left(\frac{-1}{\lambda_k} \right) [\sum_{l=1}^{L} \exp(sd) [\prod_{r=1}^{L} \frac{(s - p_{1l})}{(s - p_{1r})\,(s - p_{2r})}]|_{s = p_{1l}} \quad , & d \le 0 \\ \prod_{k=1}^{L} \left(\frac{-1}{\lambda_k} \right) [\sum_{l=1}^{L} \exp(sd) [\prod_{r=1}^{L} \frac{(s - p_{2l})}{(s - p_{1r})\,(s - p_{2r})}]|_{s = p_{2l}} \quad , & d \ge 0 \end{cases} \tag{28}$$

IV. Pairwise Error Event Probability

Since the pairwise error probability $P(c \to \hat{c})$ is defined as the probability that $D \le 0$, then one can find it by integrating the pdf, $P_D(d)$, given by eq. (28), from $-\infty$ *to* 0

$$\begin{aligned} P(c \to \hat{c}) &= \int_{-\infty}^{0} p_D(d) \quad dd \\ &= \prod_{k=1}^{L} \left(\frac{-1}{\lambda_k} \right) [\sum_{l=1}^{L} [\prod_{r=1}^{L} \frac{(s - p_{1l})}{s(s - p_{1r})\,(s - p_{2r})}]|_{s = p_{1l}}] \end{aligned} \tag{29}$$

V. Bit Error Probability

Since in nearly all applications, we are interested in the overall bit error event probability, an approximation to the bit error probability can be obtained by summing the pairwise error event probabilities as

$$P_b \approx \sum_{c \ne c} a(c, \hat{c})\; p(c \to \hat{c}) \tag{30}$$

where $a\,(c,c)$ is the number of bit errors associated with each error event, and the summation is taken over the set of dominant (most probable) error events.

VI. Conclusions:

We have analyzed the performance of an interleaved convolutional coded BPSK with perfect channel state information and coherent detection , assuming perfect synchronization. The mobile radio channel, characterized by its exponential correlation function has been taken as an example for time dispersive correlated fading channels. Analytical results have been derived for the pairwise error event probability using the characteristic function technique. An approximate bit error probability of this coded modulation scheme is obtained by summing the probability of the dominant error events.

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