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Power Spectrum Estimation and Rational Modeling of Random Processes

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Abstract: A new technique is derived to estimate the signal power spectrum density (PSD) and hence its autcorrelation lags. Several simulation results have been given to show the results accuracy. Based on the knowledge of the correlation lags, several approaches to fit the signal or random process to rational models as to approximate the spectrum to these models transfer functions square are considered. The model coefficients results are given and commented as well to be referred with other advanced approaches in the field.

Keywords:
Signal processing, power spectrum estmation, rational models, random processes.

I. Introduction:

The power spectrum of signals and random process is very important to reveal the signal energy contents distributed over the frequency line. It is important to determine the signal identity (e.g. low pass signal etc.)) and/or characteristics as the correlation properties to design equipments in communications and digital signal processing field. In this paper, we introduce a new technique to estimate the signals spectrums. The usual technique based on the fourier analysis, called periodorgan [1], doesn't give correct spectrum but mostly smeared one. This suggested technique gives the signal spectrum irrespect to its statistical distribution. The second part of the paper is to describe the spectrum by rational model that fits its spectrum. Actually, the signal process can be described by rational model that is used in its generation from white noise process. Much of the information about the signal can be obtained through the model type, the coefficients as well as the zeros and poles of the model. The trend of the signal process or its extrapolation can be determined. A lot of these researches have been devoted to this problem of fitting the estimated spectrum by rational models in engineering econometrics, biometric, etc. [2]. Here, based on the technique of first part to estimate the spectrum or the correlations lags, different techniques are considered to find these models and its accuracy.

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In the following section, the technique to find the signal spectrum is given and also simulation results to show the accuracy of the estimated spectrum. In section III, based an the estimated correlations, several methods for finding the appropriate rational model are given with the results. Finally, the main results and conclusion is drawn in section IV.

II. Power Spectrum Estimation

Despite the importance of the signal's spectrum in many fields, there is no yet a reliable technique to find it. Much of the efforts have been devoted to fit the signal or random process to autoregressive moving average model (ARMA) and using its transfer function as representing the process spectrum, [3], however the results are not satisfied. We provide a reliable technique to find the power spectrum of signals. It is based on remark for stationary random processes that the power spectrum density S(f) of a process -y(n) can be considered as [4];

$$\lim_{\Delta \to 0} E\left[y^2(n)\right] = 2\Delta\, S(f) \qquad (1)$$

as y(n) is the process existing through the very narrow high resolution rectangle window.

$$W(f) = \begin{cases} 1 & f_0 - \frac{\Delta}{2} < |f| < f_0 + \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Thus, we established this very narrow rectangle window and passing the input process "x(n)" to obtain "y(n)" at each certain position f_0, that is swept between zero and 0.5 (normalized frequency = frequency / sampling frequency), and the spectrum level at $f = f_0$, is theoretically considered as:

$$S(f_0) = \frac{E\left[y^2(n)\right]}{2\Delta} \qquad (\Delta \approx 0) \qquad (3)$$

and $$y(n) = x(n) * w(n) \qquad (4)$$

(* denoted convolution and w(n) is the window impulse response)
"y(n)" is narrow band process and it is known as the bandwidth of the signal decreases, the time averages of the signal samples become closer to the statistical averages; so an anticipated good estimate is obtained as:

$$\hat{S}(f_0) \approx \frac{1}{2\Delta N} \sum_{n=1}^{N} y^2(n) \qquad (5)$$

This value is plotted against f_0 and the process is continued for other frequency point. To establish the very narrow rectangle window with its required high resolution, we used the digital filter:

$$H(z) = \frac{\prod_{i=1}^{3}\left(1 - Z_i Z^{-1}\right)\left(1 - Z_i^* Z^{-1}\right)}{\prod_{i=1}^{3}\left(1 - P_i Z^{-1}\right)\left(1 - P_i^* Z^{-1}\right)} \qquad (6)$$

Each corresponding pole and zero are chosen to lie on the same frequency (digital resonator). To achieve high resolution, the poles are chosen to lie at radius = 0.9999 and the zeros at radius = 0.96. The main pole-zero pair to lie at frequency angle $2\pi f_0$ and the two adjacent poles-zeros to lie at frequencies angle $2\pi\left(f_0 \pm 3\times10^{-6}\right)$ rad/sec. The bandwidth of the window is found 17×10^{-6} HZ (i.e. Δ = 17 $\times$ 10^{-6}). As in (2), the filter transfer function is normalized to have maximum magnitude equal to one. Then, it is normalized once again to give equal amplitude to sinusoidal signal at its center frequency. However the spectrum level stays to need more accurate scaling to adjust its level to the true signal power level. In the coming results, both exact and estimated spectrums are normalized to 0 dB value at f = 0 to be able to held the comparison. The performance of this technique is shown through three examples

Example 1: For rational model (ARMA(4,4))

$$x(n) + 1.72\, x(n-2) + 0.81\, x(n-4) = \epsilon(n) + 1.23\, \epsilon(n-2) - 0.245\, \epsilon(n-4) \qquad (7)$$

is driven by gaussian white noise "$\epsilon(n)$" to obtain x(n). The white noise is obtained as sum of 1000 independent runs of IMSL subroutine to generate such noise process. The spectrum of this input is flat. The spectrum of x(n) is shown in Fig. (1) as the estimated spectrum is compared with the exact spectrum of the above model to show the accuracy of the spectrum estimation. The plot is for 100 points.

Example 2: Minimum phase ARMA (4,3), model

$$x(n) - 2.202\, x(n-1) + 2.628\, x(n-2) - 1.835\, x(n-3) + 0.731\, x(n-4) = \epsilon(n) + 1.5\, \epsilon(n-1) + 1.05\, \epsilon(n-2) + 0.392\, \epsilon(n-3) \qquad (8)$$

The zeros of the model as well as the poles lie inside unit circle. The resultant spectrum estimate is in Fig.(2).

Example 3: Moving average model; MA (3)

$$x(n) = \epsilon(n) + 1.5\, \epsilon(n-1) + 1.05\, \epsilon(n-2) + 0.392\, \epsilon(n-3) \qquad (9)$$

The results are in Fig. (3). These simulated examples prove the advance and reliability of this spectrum estimator to yield good faithful estimate of the underlying process spectrum. The power spectrum density is known to be given by:

$$S(\omega)=r(0)+2\sum_{m=1}^{\infty}r(m)\ \cos\ m\omega \qquad (10)$$

as r(m) is the m^{th} lag correlation $\left(r(m)=r(-m)\right)$

Thus, the autocorrelation lags can be obtained as

$$r(m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}S(\omega)\cos m\omega\ d\omega \qquad (11)$$

This estimation procedure of random signal process correlation properties can have several implications and advantages in signal processing applications [5]. The following section shows the use of these correlation lags to fit the signal process to rational model.

III. Rational Modeling

Two approaches are basically derived; the first is based on the ergodicity property (as assumption of, the statistical correlations equal to the time correlations average) to derive a set of linear equations to find the coefficients of the model. The second approach considered is based on fitting the rational model to the estimated power spectrum. We show both approaches and their results.

1- Autogressive (AR) part of ARMA model

The generated random process by this model is given by:

$$x(n)+\sum_{k=1}^{p}a_k\,x(n-k)=\sum_{k=0}^{q}b_k \in (n-k) \qquad (12)$$

as (p,q) is the model order and $\in(n)$ is the driving white noise.

The spectrum is represented as

$$S(\omega)=\left|\frac{\sum_{k=0}^{q}b_k\,e^{-j\omega k}}{1+\sum_{k=1}^{p}a_k\,e^{-j\omega k}}\right|^2 \qquad (13)$$

If equation (12) is multiplied by x(n-m) and the statistical average operator is performed, we get the set of equations:

$$r(m)+\sum_{k=1}^{p}a_k\,r(m-k)=0 \qquad \text{as } m\ \geq q+1 \qquad (14)$$

r(m) = E [x(n) x(n-m)]

$E \underset{=}{\Delta}$ statistical average

These overdetermined equations can be represented in matrix form as

$$RA = 0 \qquad (15)$$

where $A^T = [1 \quad a_1 \quad a_2 \; \ldots\ldots \; a_p]$

and $$R = \begin{bmatrix} r(q+1) & r(q) & \cdots & r(q+1-p) \\ \vdots & & & \\ r(q+t) & & \cdots & r(q+t-p) \end{bmatrix}$$

This technique is first given in [2]. To solve this set of eqs., the problem is considered as (minimization problem):

$$\min A^T R^T R A \tag{16}$$

Two ways can be throught to find the optimal vector A. The first way to find A as the minimum eigenvalue vector of the matrix $R^T R$, then the elements of the vector are scaled to give one to the first element. The second way to consider the constraint that the first element of A is restricted to one, so the problem in (16) is actually:

$$A^T R^T R A = \overline{A}^T H \overline{A} + 2b^T \overline{A} + C \tag{17}$$

where

$$\overline{A}^T = [a_1 \; a_2 \; \ldots . a_p]$$

and C = first element (1,1) of $R^T R$

b^T = the rest of first line of $R^T R$

H = the corner matrix after cancelling the first column and row of $R^T R$ This way gives the solution:

$$A^T = -H^{-1} b \tag{18}$$

The second approach based on fitting the model to the estimated spectrum gives the equations: (based on eq. (13))

$$\sum_{k=0}^{P} \sum_{\ell=0}^{P} a_k a_\ell S(\omega) e^{-j(k-\ell)} = \sum_{k=0}^{q} \sum_{\ell=0}^{q} b_k b_\ell e^{-j(k-\ell)} \tag{19}$$

or $$\sum_{k=0}^{P} \sum_{\ell=0}^{P} a_k a_\ell \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{j(m-k+\ell)\omega} \, d\omega = 0 \qquad \text{as } m \geq q+1 \tag{20}$$

$(a_0 = 1)$

or for m^{th} lag, we get $$A^T R_m A = 0 \tag{21}$$

as $$R_m = \begin{bmatrix} r(m) & r(m-1) & \cdots\cdots & r(m-p) \\ & & & \\ r(m+p) & & \cdots\cdots & r(m) \end{bmatrix}$$

Considering all the correlation lags possible form (positive and negative), we get the problem formulation:

$$\min \quad A^T \left\{ \sum_{m=q+1}^{t} \left(R_m + R_m^T\right) \right\} A \tag{22}$$

Similar to the problem in (16), we have two ways to solve the problem in (21) as the minimum eigenvalue problem solver or quadratic problem as in (17). Therefore, we have the total of four methods to find A, the corresponding results for specific example is next considered.

Example: the following minimum phase ARMA (4,3) model is used to obtain random process that is driven by gaussian white process

$$H(z)=\frac{1+1.5Z^{-1}+1.05Z^{-2}+0.392Z^{-3}}{1-2.202Z^{-1}+2.628Z^{-2}-1.835Z^{-3}+0.731Z^{-4}} \tag{23}$$

The spectrum estimation of this process is shown in Fig. (2). For the overdetermined system equation, up to 22 equations are considered in the 4th method to find A. The obtained results are in table I:

Table I

Method	$a_1 = -2.202$	$a_2 = 2.628$	$a_3 = -1.835$	$a_4 = 0.731$
I	-2.584	3.122	-1.946	0.586
II	-2.248	2.397	-1.360	0.419
III	-2.487	3.303	-2.477	0.994
IV	-2.316	2.847	-2.039	0.846

It is clear the best results are obtained with last method (IV). The same examples are repeated for t=42, i.e. more overdetermined equations. However, the results accuracy compared with Table I. were less. We understand that adding more information can lead to better results to this sensitive problem, but it seems the spectrum estimate accuracy or its associated error limits the problem exact solution with these different assumptions based techniques.

2- AR-model:

For this particular model, the signal random process is generated according to:

$$x(n)+\sum_{k=1}^{P} a_k\, x(n-k)=b_0 \in(n) \tag{24}$$

and the spectrum is given by:

$$S(\omega)=\frac{|b_0|^2}{\left|1+\sum_{k=1}^{P} a_k\, e^{-j\omega k}\right|^2} \tag{25}$$

The known Yule-Walker method to estimate AR-parameters is based on the existing of ergodic process that is known as solving the equations set:

$$RA=|b_0|^2\, C \tag{26}$$

where $A^T = [1 \quad a_1 \,\ldots\ldots\, a_p]$
$C^T = [1 \quad 0 \,\ldots\ldots\, 0]$

and $$R = \begin{bmatrix} r(0) & r(1) & \cdots\cdots & r(-p) \\ r(1) & r(0) & & \\ & & & \\ r(p) & \cdots\cdots & \cdots\cdots & r(0) \end{bmatrix}$$

The second approach based on sitting the spectrum by the model gives similar problem to (22) as:

$$\min \; A^T \left\{ \sum_{m=1}^{t} \left(R_m + R_m^T \right) \right\} A \qquad (27)$$

that is solved similar to method IV.
Example: the following AR(4)-model is considered

$$H(z) = \frac{1}{1 - 2.202Z^{-1} + 2.628Z^{-2} - 1.835Z^{-3} + 0.731Z^{-4}} \qquad (28)$$

the estimated coefficients are shown in table II

Table II

Method	$a_1 = -2.202$	$a_2 = 2.628$	$a_3 = -1.835$	$a_4 = 0.731$
I	-2.182	2.568	-1.768	0.688
II	-2.662	2.692	-1.609	0.184

The results obtained with Yule-Walker method is more accurate.

3- Moving - average (MA) model:
The underlying model process is given by:

$$x(n) = \sum_{k=0}^{q} b_k \in (n-k) \qquad (29)$$

where $\in(n)$ is the driving input gaussian while process (unit variance).
The statistical average correlations are given by:

$$r_x(n) = \begin{cases} \sum_{k=0}^{q} b_k b_{x-n} & -q \le n \le q \\ 0 & \text{otherwise} \end{cases} \qquad (30)$$

The spectrum is given by:

$$S(\omega) = \left| \sum_{k=0}^{q} b_k e^{-j\omega k} \right|^2 \qquad (31)$$

Example: The following MA (3)-model is used

$$x(n) = \in(n) + 1.5 \in(n) + 1.05 \in(n-2) + 0.392 \in(n-3) \quad (32)$$

The process spectrum is shown in Fig. (3). The autocorrelation lags for this spectrum are calculated. We found the correlation lags for n > q, are not vanished as expected to the statistical correlations as in (30). The value of r(q+m) is comparable by r(q) for certain small m. However, we tried to use the estimated correlation results with the statistical average assumption to obtain the model coefficients as:

$$\min \left\{ \sum_{n=0}^{q} \left(f(n) - r(n)\right)^2 \right\} \quad (33)$$

as $$f(n) = \sum_{k=0}^{q} b_k b_{k-n} \qquad (q = 3)$$

The approach based on fitting the estimated spectrum by the model theoretical spectrum is also considered as the problem solution of;

$$\min \left\{ \sum_{i=1}^{N} \left(S(\omega_i) - \hat{S}(\omega_i)\right)^2 \right\} \quad (34)$$

where $S(\omega)$ as in (31) and $\hat{S}(\omega_i)$ is the spectrum in Fig. (3). (N = 100). The results of the two problems in (33) & (34) respectively are:

Table III

Method	a_1 = -2.202	a_2 = 2.628	a_3 = -1.835
I	1.16	0.756	0.392
II	1.391	0.895	0.504

The results of the second method are more accurate. It is our belief that the fitting spectrum by the rational model spectrum should be more accurate results, despite its need to large number of points to represent the spectrum. The spectrum estimator can provide as much as 30,000 points, however the computer load is largely demanded. In all the results, adding noise to the random process means distorting the spectrum estimate and less accurate coefficients estimation. The rational modeling of random process is still active area in research to introduce more advanced fitting techniques that are striving for increased accuracy [3], [6], [7].

IV. Conclusion:

A new technique to estimate the power spectrum density of random signals is given and its results are shown. Hence, the signal correlation lags properties can be obtained, a crucial step for signal processing applications. Then, the modeling of the signal process by rational model driven by white noise is considered. The results for the different rational models are presented for two approaches to find its accuracy and the justifications of the different assumptions that are based on. The direct spectrum estimate modeling is more realistic than the approach based on the ergodicity random process property assumption. To improve the models coefficients accuracy for the given spectrum estimator accuracy, we are still looking for more advanced techniques as well as in the research medium.

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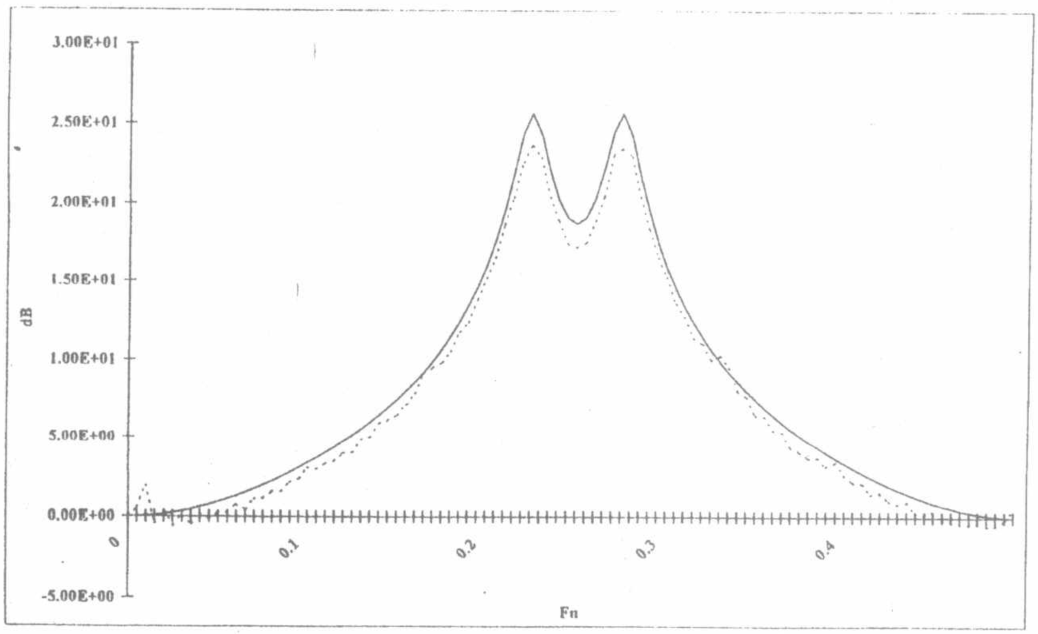

Fig. (1): Spectrum vs. normalized frequency
" --------" estimated and "________" exact

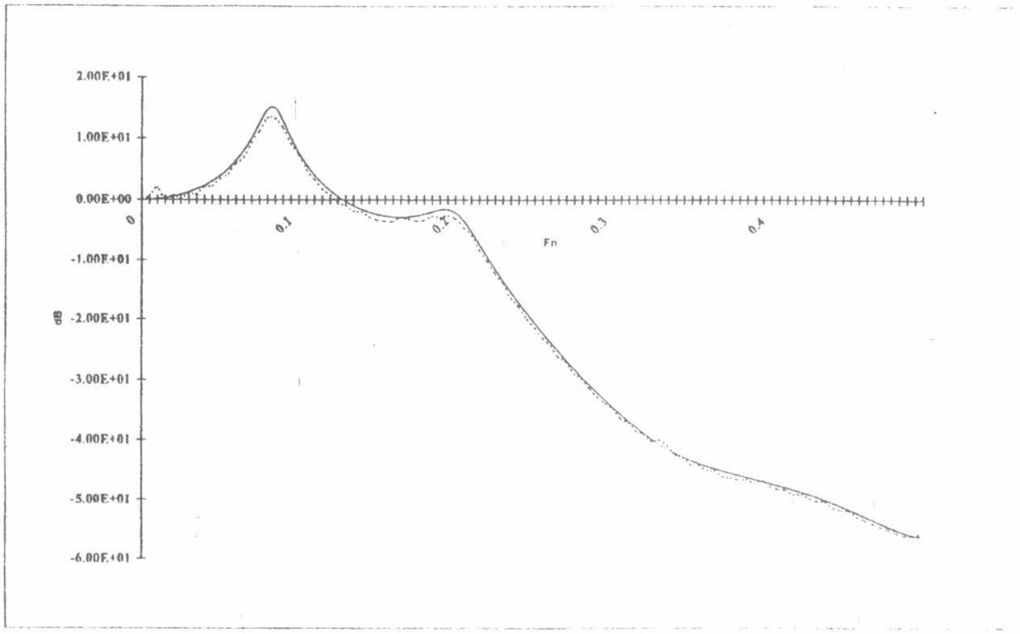

Fig. (2): Spectrum vs. normalized frequency
" --------" estimated and "________" exact

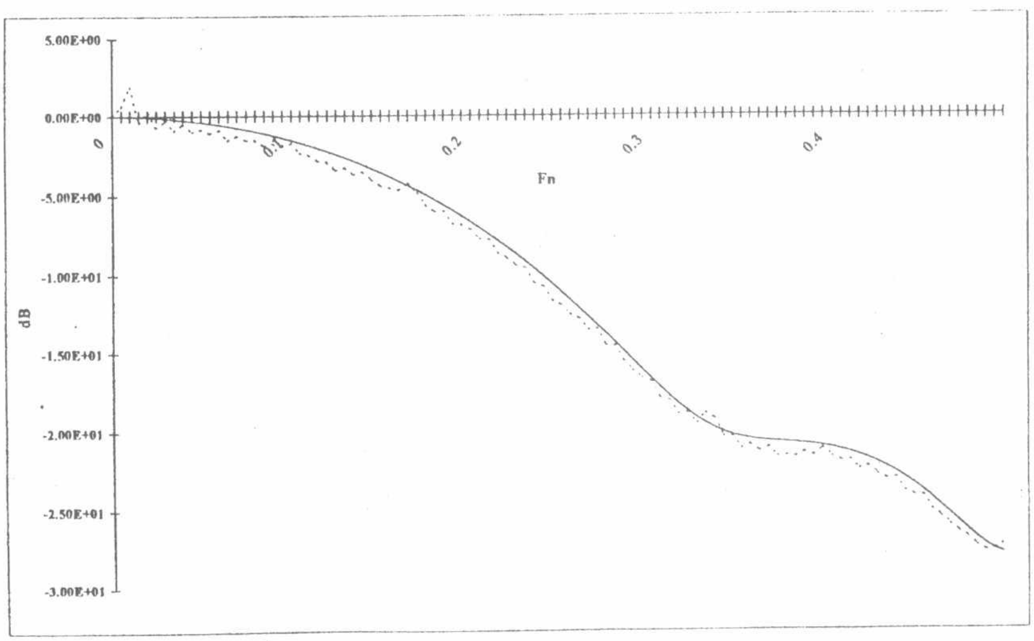

Fig. (3): Spectrum vs. normalized frequency
" ______" estimated and "______" exact