Different Models of Random Walk

Afrah Al-Bossly[1, *], Bashayer Salim Aldbalin[1], Gazwa Ibrahim Alotaibi[1], Rahaf Dobian Alsubaie[1], Rawan Diab Althaqafi[1]

[1] *Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia.*

A B S T R A C T

This paper is devoted to studying what call irregular random walk on the non-negative integers. The random walk plays an essential role in different areas of applications. It models many problems in a way to make it easier to handle and study. A random walk, in general, is a sequence of partial sums of independent identically distributed random variables. The type which well be considered in is partial sums of random variables that are not necessarily independent or identically distributed. The type of random walk of being transient or recurrent is one of the most important concepts to be studied. The random walk is said to be recurrent if the walker starts at some state and the return time to the starting state is finite almost surely, and transient if there is a positive probability of not returning to the starting state. The recurrent random walk is positive if the mean return time is also finite, and it is null recurrent otherwise.

1 . Introduction

The irregular random walk is on the non-negative integersZ^+, [1,2]. Particle moves on Z^+ starting at 0 and each unit of time it makes either a jump of size m to the right with probability p or goes back one-unit distance with probability$q = 1 - p$. In which case, the summands of the partial sums are not necessarily independent or identically distributed random variables [3, 4], this is the reason for calling Markov chain is irregular random walk [5]. It was proven that the $(m, 1)$IRW is positive recurrent if and only if $< \frac{1}{m+1}$, transient if and only if $p > \frac{1}{m+1}$, and null recurrent if and only if $p = \frac{1}{m+1}$[5]. Another irregular random walk is obtained if the particle can make one-unit distance jump to the right with probability p or to make a jump of size n to the left with probability $1 - p$ [6]. Also it is proven that the $(1, n)$IRW is positive recurrent if and only if $p < \frac{n}{1+n}$, transient if and only if $p > \frac{n}{1+n}$, and null recurrent if and only if $p = \frac{n}{1+n}$, these two models are generalized by allowing the jump to the right to be of size m and to the left to be of size n[7].

A different type of irregularity of random walks was studied. A particle moves on Z^+ starting at 0 and makes either a

* Afrah Al-Bossly, College of Science and Humanities, Prince Sattam Bin Abdulaziz University, Saudi Arabia, +9665486809291& a.basli@psau.edu.sa

jump of size m to the right with probability p or goes back (falls) straight to 0 with probability $q = 1 - p$. We show that this Markov chain is positive recurrent regardless of the values of m and p [8] Also, irregular random walk with varying probabilities, p_k is considered. In this section we will present some of the definitions and concepts of probability theory in general and Markov chains. If we have probability space $(\Omega, £, p)$ and a set $X = \{x_t : t\epsilon T\}$ of random variables (Measurable), knowledge of the sample space Ω and take their values out of space S, the set X it's called (Random Process $\equiv$ Stochastic Process) and S is called(State Space) for random operation, and the set T(Parameter Space$\equiv$ Index Set).Many random processes check the following property which is called the (Markov property) that the future values of the process are not affected by the past values if the current value of the process is known [9].

2 . Chosen Models of Random Walk

2.1 . Irregular Random Walk on Non-Negative Integers Z^+

Remember that the random walk is a sequence of consecutive partial totals of random variables following the same probability distribution, and our study in this section of the irregular random walk on the line of non-negative integers Z^+. Where the particle of motion starts from position 0 and then makes a step along the

length of m of the unit of distance to the right with probability p, or a step along the length of n the unit of distance to the left with probability $q = 1 - p$, when the particle is less than n unit from state 0, its next step to the left is zero. this leads to the fact that the length of the step to the left depends on the distance of the point from state 0, which in turn leads to random variables that are sequential partial totals are random variables that are not independent and do not follow the same probability distribution [10,11].

This is why the Markov chain we are studying is called irregular random walk on non-negative integersZ^+, in this case, we have Markov chain state space is the class of non-negative integers Z^+ and the probability of transition is:

$$\begin{cases} P(i, i + m) = p, & i \geq 0 \\ P(i, i - n) = q = 1 - p, & i \geq n \\ P(i, 0) = 1 - p, & i < n \\ P(i, k) = 0, & k \neq i + m, i - n, 0 \end{cases} \quad (1)$$

Remember that if the irregular random walk (IRW) is zero starts and the time of return is (finite), the irregular random walk (IRW) is recurrent, while irregular random walk (IRW) is transient if there is a positive probability of not returning to state zero. we will denote this model of a random walk with the symbol (, n)IRW .The following is a Probability-generating function that can be used to infer the necessary and enough condition for the irregular random walk (IRW) to be positive recurrent. The irregular random walk (IRW) is Aperiodic and irreducible.

2.1.1 . Definition, [12].

The Probability generating function and corresponding probability distribution $\pi = (\pi_k : k \geq 0)$ is defined as the following formula:

$$\Pi(S) = \sum_{k=0}^{\infty} \pi_k S^k \quad (2)$$

The following theories play an important role in proving the validity of some of the results in this section.

2.1.2 . Theory

Markov chain irreducible has a stationary distribution if and only if the chain is a positive recurrent

2.1.3 . Theory

If the Markov chain is irreducible, the state space S and zero are (Reference state) and $Q = (P(x, y), x, y \in S\backslash\{0\})$, and the Markov chain are transient if and only if the system is found:

$QX = X, \quad 0 \leq x_i \leq 1, \quad x_i \in S\backslash\{0\}$a nonzero solution.

2.1.4 . Theory

Irregular random walk$(1, n)IRW$ is:

Positive recurrent if and only if $p < \frac{n}{1+n}$

Transient if and only if $p > \frac{n}{1+n}$

Null recurrent if and only if $p = \frac{n}{1+n}$

2.1.5 . Theory

Irregular random walk$(m, 1)$IRW is:

- Positive recurrent if and only if $p < \frac{1}{m+1}$

- Transient if and only if $p > \frac{1}{m+1}$

- Null recurrent if and only if$p = \frac{1}{m+1}$

2.1.6 . Theory

Irregular random walk(m, n)IRW is:

- Positive recurrent if and only if $p < \frac{n}{n+m}$

- Transient if and only if $p > \frac{n}{n+m}$

- Null recurrent if and only if $p = \frac{n}{n+m}$

2.2 . Site-Dependent Irregular Random Walk,[13].

In the following section it will be examine the irregular random walk (IRW)of the on Z^+ a starting g of state 0 so that the probability of moving from state k to state $k + m$ is p_k, and the probability of moving to the state $k - n$ is $1 - p_k$ that is, moving right and left depends on state where you move from.

We will start by examining the case in which it is $n = 1$, m=2, the probability of transition is as follows:

$$\begin{cases} P(X_{j+1} = k + 2 | X_j = k) = p_k, & k \geq 0 \\ P(X_{j+1} = k - 1 | X_j = k) = 1 - p_k, & k \geq 1 \\ P(X_{j+1} = 0 | X_j = 0) = 1 - p_0 \end{cases} \quad (3)$$

This type of Markov chain with the symbol is marking by$(2,1)IRW$.

2.2.1 . Theory, [13.

As a randomized random walk (IRW) on non-negative integers, Z^+ site-dependent and knowledge of equation (4), the necessary condition for a probability distribution π to be a stationary distribution is:

$$3\sum_{k=0}^{\infty} \pi_k p_k + \pi_0 q_0 = 1, \quad q_0 = 1 - p_0 \quad (4)$$

2.3 . Site-Dependent Irregular Random Walk,.[14].

In this section we will examine a generalization of what was previously studied in the previous chapter, where the particle is allowed at state k to make a step to the right along the m length unit with probabilityp_k, or a step to the left along the length unit with probability $q_k = 1 - p_k$, this type of irregular random walk and depends on the state called (m,1)Site-Dependent Irregular Random Walk, Symbolized by the symbol $(m, 1)SDIRW$, the probability of transition is as follows:

$$\begin{cases} P(k, k + m) = p_k, & k \geq 0 \\ P(k, k - 1) = 1 - p_k, & k \geq 1 \\ P(0, 0) = 1 - p_0 \end{cases} \quad (5)$$

As a result of equation (4) it is clear that $(m, 1)SDIRW$ is irreducible.

The following theory represents a generalization of theory (2.1.1) where the particle can take steps along the length of $m, 1$.

2.3.1. Theory

Given the random walk $(m, 1)SDIRW$ defined by equation (6), the necessary condition for probability distribution π is to be a stationary distribution:

$$\sum_{k=0}^{\infty} \pi_0 p_0 = \frac{1 - \pi_0 q_0}{m+1}, \quad q_0 = 1 - p_0 \qquad (6)$$

2.4. Another model of the irregular random walk on non-negative integer, [14].

In this section we will consider a particle that makes each plate a step time by the length of m, a unit of distance to the right with probability p, or in the opposite direction, making a step to zero with the probability of $q = 1 - p$. In this case, the length of the step made by the particle depends on the distance from zero. If S_n denotes the position of the particle after n time unit, S_n is the sequence of the partial totals of random variables that are independent and do not follow the same probability distribution. This model is unusual for a random walk and is another model of an irregular random walk.

In this case, the state space is a set of non-negative integers that are accepted by the division of $m, mZ^+, \ m = 0,1,2,...$In this case, zero will be regarded as (Retaining Barrier) as we will explain later that there is another type of irregular random walk and where the probability depends on where the jump is made. This sequence p depends on state k. in these cases, we will show that in the first case, the irregular random walk is positive recurrent, regardless of the probability value p and the length of the jump m.

We will also prove that in the second case, for irregular random walk transient we would assume that $q_k = 1 - p_k$ converges to zero quickly enough. We will also use the generator function to prove that the movement is positive in the case of constant probability p.the stationary distribution of the Markov chain is the case of π and $\pi = (\pi(x): x \geq 0)$ has its properties:

$$* \pi(x) \geq 0, \quad ** \sum_x \pi(x) = 1, \quad *** \pi p = \pi$$

Where it refers to the transition matrix of the Markov chain, the probability generator function is as follows:

$$\Pi(S) = \sum_{k=1}^{\infty} \pi_k S^k \qquad (7)$$

2.5. Irregular random walks (IRW) with constant jump probability

As a Markov chain whose state space mZ^+ and probability transition are:

$$\begin{cases} P(i, i+m) = p, & km \geq 0 \\ P(i, 0) = 1 - p, & km \geq 0 \end{cases} \qquad (8)$$

These Markov chains are called (IRW) on mZ^+.

From the definition, we note that the 0 is a retaining barrier in the sense that it allows the particle to remain in the zero states for some time.

The following theory illustrates the positive recurrent of the Markov chain defined by equation (8).

2.5.1. Theory

Irregular random walk (IRW) defined by equation (8) on mZ^+ is positive recurrent.

Proof

To prove that; the random Walk and knowledge of the equation (8) is positive recurrent enough to verify the presence of stationary distribution π.

of the equation: $\qquad \pi p = \pi$

get $\pi_y = \sum_x \pi_x p_{xy}, \quad y \in Z^+$

and from them, we get $\pi_0 = \sum_k \pi_{km} p_{km,0}$

$$= \sum_k \pi_{km}(1 - p)$$

$$\Rightarrow \pi_0 = (1 - p) \sum_k \pi_{km}$$

$$= (1 - p)(1)$$

$$= (1 - p) \qquad (9)$$

and we also get.

$$\pi_{km} = \pi_{(k-1)m} p_{(k-1)m,km}$$

$$= p\pi_{(k-1)m}$$

$$= p^k \pi_0$$

$$\Rightarrow \pi_{km} = (1 - p)p^k, \quad k = 0,1,2,..... \qquad (10)$$

It is a stationary distribution of the chain in question, and since the chain is irreducible, it must be a positive recurrent.

2.6 Irregular Random Walk on Non-Negative Integers with

2.6.1. Varying Probabilities, [15].

In this section we will consider the sequential possibilities $\{p_k, k \geq 0\}$ where $0 < p_k < 1$, P_k indicates the probability of moving from state k to $k + m$ and $1 - p_k$ indicates the probability of moving from state k to zero. Constant probability p as in the previous section, Markov chain on mZ^+ and the probability of transition is:

$$\begin{cases} P(X_{i+1} = (k+1)m | X_i = km) = p_{km} \\ \quad P(X_{i+1} = 0 | X_i = km) = 1 - p_{km} \end{cases} \qquad (11)$$

This chain is irreducible, and if we consider the particle starts its motion from state 0 and that T_0 indicates the first time back to state 0 and $m_0 = E(T_0)$ we find that:

$$P(T_0 = n + 1) = p_0 p_m p_{2m} \ldots \ldots p_{(n-1)m}(1 - p_{nm}) \quad (12)$$

and whereas

$$P(T_0 < \infty) = \sum_{n=0}^{\infty} P(T_0 = n + 1) \quad (13)$$

$$P(T_0 < \infty) = (1 - p_0)$$

$$+ \lim_{N \to \infty} \sum_{n=1}^{N} p_0 p_m p_{2m} \ldots p_{(n-1)m}(1 - p_{nm})$$

$$= 1 - \lim_{N \to \infty} \prod_{k=0}^{N} p_{km} \quad (14)$$

2.7 . Random walk and its application to solution of heat, [16].

Methods in Monte Carlo: The Monte Carlo methods that are introduced in [16], this study is used Semi Floating .Random Walk (SFRW) as shown in figure1, and Full Floating Random Walk (FFRW) to define nature in the domain of the meaning (See Figure 2)
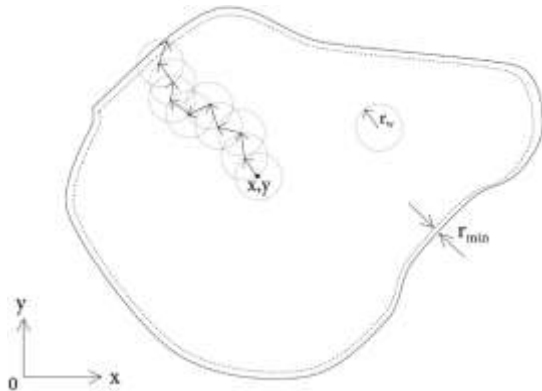

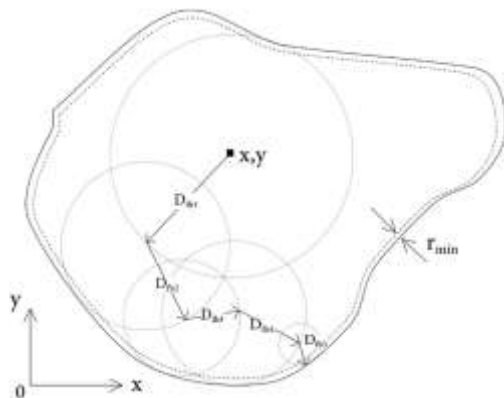**Figure 1: Typical Semi Floating Random Walk (SFRW) [16].**

Figure 2: Typical full floating random walk [16].

Fixed Random Walk (FRW) method is realistic to solve transient heat conduction equation as shown in Figure 3.
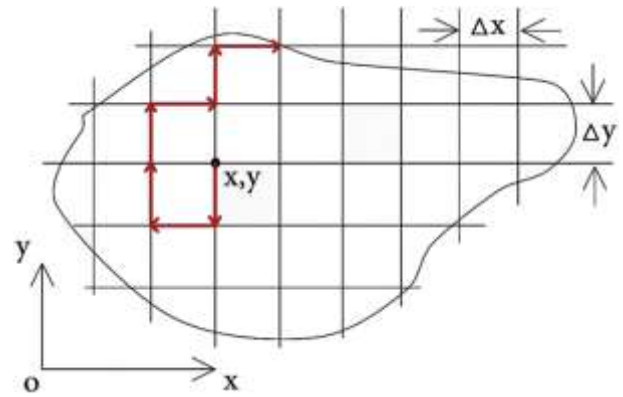

Figure 3: Typical fixed random walk. [16].

It was detected that FFRW method is the fastest method to stable state conditions and FRW method has ability to run both stable state and transient heat conduction equation. It was showed that the results are in good agreement with those of the finite difference method, [16].

3 . Conclusion

The irregular random walk (IRW) defined by equation (9) is a positive recurrent on mZ^+ if and if only i

$$\lim_{N \to \infty} \prod_{k=1}^{N} p_{km} = 0 \quad (15)$$

Meaning that if and if only the chain was:

$$\sum_{k=1}^{\infty} 1 - p_{km} \quad (16)$$

Spacing and that because:

$$\sum_{k} 1 - p_{km} = \infty \Leftrightarrow \lim_{N} \prod_{k=1}^{N} p_{km} = 0 \quad (17)$$

In continue, different boundary conditions were examined and for each technique were applied discretely.

References

[1] David Harel and Yehuda Koren, "*Clustering spatial data using random walks*." In Knowledge Discovery and Data Mining (KDD'01), 281-286, 2001. .https://dl.acm.org/citation.cfm?id=502552

[2] Marcel F. Neuts"An Interesting Random Walk on the Non-Negative Integers"*Journal of Applied Probability* 31(1), 48-58 Mar., 1994. https://doi.org/10.2307/3215234

[3] Rabi N. Bhattacharya and Edward C. Waymire "Stochastic Processes with Applications", 2009. https://www.ime.usp.br/~fmachado/MAE5709/SPABhattacharya.pdf

[4] Francois Fouss, Alain Pirotte, Jean-Michel Renders, and Marco Saerens, Random- walk computation of similarities between nodes of a graph with application to collaborative recommendation. IEEE Transactions on Knowledge and Data Engi- neering, 19(3), 355-369, 2007. https://ieeexplore.ieee.org/document/4072747

[5] David Aldous and James Allen Fill "Reversible Markov Chains and Random Walks on Graphs٢٠١٤ ".https://www.stat.berkeley.edu/~aldous/RWG/book.pdf

[6] K. L. Chung, Elementary Probability Theory with Stochastic Processes, Springet-Verlag, Berlim, 1974. 1.https://www.springer.com/gp/book/9781475739732

[7] W. Feller, An Introduction to Probability Theory and its Applications, Vol. I, 3 rdedition, John Wiley; Sons, New York, 1968. https://www.wiley.com/en-us/An+Introduction+to+Probability+Theory+and+Its+Applications%2C+Volume+1%2C+3rd+Edition-p-9780471257080

[8] P.G. Hoel,S. C. Port, and C. J. Stone, Introduction to Stochastic Processes, Houghton Mifflin Company, Boston, 1972. http://www.mat.ufrgs.br/~giacomo/Livros/Hoel-Port-Stone-Stochastic.pdf

[9] Mokhtar Konsowa and Joanna Mitro, "*The Type Problem of Random Walks on Trees"*, J. of Theoretical Probability. 4(3), 535-550, 1991. https://link.springer.com/article/10.1007/BF01210323

[10] Mokhtar Konsowa and Hassan Okasha, *Irregular Random Walks on Non-Negative Integers*, Kyungpook Mathematical Journal, 40(2), 431-436, 2000. https://www.researchgate.net/publication/263399254_Irregular_Random_Walk_on_Noonegetive_Integers

[11] Sidney Resnick, *Adventures in Stochastic Processes*, Birkhauser, Boston, 1994. https://play.google.com/store/books/details?id=VQrpBwAAQBAJ&rdid=book-VQrpBwAAQBAJ&rdot=1&source=gbs_vpt_read&pcampaignid=books_booksearch_viewport

[12] Pierre Br´emaud,Basic Markov Chains, 2015. https://www.di.ens.fr/~busic/cours/M2amis/markov.pdf

[13] MARTIN L. PUTERMAN Markov Decision Processes Discrete Stochastic Dynamic Programming, John Wiley & Sons, Inc., 2005. https://dl.acm.org/citation.cfm?id=528623

[14] Konsowa, Mokhtar-H and Okasha, Hassan-M "Site- Dependent Irregular Random Walk on Nonnegative Integers" Journal of the Korean Statistical Society, 32 (4) 401-409, 2003. http://www.koreascience.or.kr/article/JAKO200311922066700.page

[15] Douglas C. Montgomery and George C. Runger, "Applied Statisticsand Probability for Engineers" 2011 http://www.um.edu.ar/math/montgomery.pdf

[16] S. Talebi*, K. Gharehbash, H.R. Jalali "Study on random walk and its application to solution of heat conduction equation by Monte Carlo method"*Progress in Nuclear Energy96, 18-35, 2017.* http://dx.doi.org/10.1016/j.pnucene.2016.12.004