Al-Azhar Bull. Sci. Vol. 22, No. 2 (Dec.): pp. 9-20, 2011.

COSMOLOGICAL MODEL WITH ULTRARELATIVISTIC IDEAL GAS USING GRAVITATIONAL AND THERMO–MECHANICAL HAMILTONIAM FORMULATION

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Abstract

A brief review of the Hamiltonian theory of self gravitating perfect fluid, which has been established by Kijowski et al (1990), has been discussed in this work. The formulation of the spherically symmetric cosmological problem has been derived. The most general 3-dimensional metric in the case of spherically symmetric space-time has been considered. In addition, the parameters which govern the dynamics have been fixed. The dynamical equations have been derived. The problem of homogeneous Universe has been considered for the Ultrarelativistic ideal gas. The dynamical equations have been derived. Three analytical solutions have been obtained.

Key words: Hamiltonian theory-spherically cosmological Model-Ultrarelativistic ideal gas.

1. Hamiltonian formulation of general relativity

To investigate the Hamiltonian formulation of general relativity ([1],[2]) we can proceed as follows. Let M be the space-time manifold with coordinates (x^μ); μ = 0,1,2,3 and let $\Sigma \in M$ be the three-dimensional initial-value surface. Cauchy data for the free gravitational field are described by the Riemannian metric g_{kl} on Σ (Latin indices run from 1 to 3) and by the so called ADM momentum P^{kl}, where

$$P^{kl} = \sqrt{\det g}\,\left(K\,g^{kl} - K^{kl}\right) \tag{1.1}$$

K_{kl} is the extrinsic curvature tensor of Σ and $K \equiv K_{kl}\; g^{kl}$.

Four of ten Einstein equations do not contain time derivative of P^{kl} and g_{kl} namely

$$R(g) - \frac{1}{\det g}\left(P^{kl}\,P_{kl} - \tfrac{1}{2}P^2\right) = 0 \tag{1.2}$$

where $P \equiv g_{kl}\; P^{kl}$, and

$$P_{k\;|\;l}^{l} = 0 \tag{1.3}$$

where "$|$" denotes the covariant derivative on Σ with respect to the metric connection generated by g. These 4 equations can be treated as Hamiltonian constrains for the Hamiltonian system $\left(P^{kl}\ ,\ g_{kl}\right)$. There are 4(per point) "Lagrange multipliers" N (lapse function) and N^k (shift vector), canonically conjugate to the four constraints [2]. The dynamics of the data $\left(P^{kl}\ ,\ g_{kl}\right)$ resulting from the remaining 6 Einstein equations which can not be uniquely defined unless the parameters $(N,\ N^k)$ are described at each point of Σ and at each instant of time $t = x^0$.

The formulation of the general relativity coupled to hydro-dynamics has been proposed by Kijowski, Smolski and Gornicka (see [3] - we will refer to this paper by KSG). It shows that the phase space, describing both gravitational and thermo-mechanical degrees of freedom (2+4 per point), can be described by the same mutually conjugate objects $\left(P^{kl}\ ,\ g_{kl}\right)$, as in the vacuum case. Zero on the right hand side of equations (1.2) and (1.3) is replaced by corresponding components of the matter-energy-momentum tensor. However, the equations can no longer be considered as constraints. They enable us to calculate uniquely the lapse and the shift in terms of the data $\left(P^{kl}\ ,\ g_{kl}\right)$. Finally, the time evolution of the system is uniquely generated by a regular, non-constrained Hamiltonian $H = H\left(P^{kl}\ ,\ g_{kl}\right)$.

The goal of "KSG" investigation was a construction of a non-constrained Hamiltonian $H = H\left(P^{kl}\ ,\ g_{kl}\right)$, such that the evolution equations generated by H are precisely the Einstein-Euler equations for the self-gravitating perfect fluid whose mechanical and thermo-dynamical properties are described by [4].

$$S = S\left(\mathrm{e}, V\right) \tag{1.4}$$

where S is the molar entropy, *e* is the internal energy per mole and *V* is the molar volume.

In the absence of gravitational interaction, the "KSG" suggest the following treatment of the hydrodynamics as field theory. The 3-dimensional material space *Z* is considered with an appropriate geometric structure. Points of *Z* correspond to particles of the matter. The configuration can represented as mapping $\varsigma\ : M \longrightarrow Z$.

Given a coordinate system (z^a), *a* =1, 2, 3; the mapping is described by three functions $z^a = \varsigma^a\left(x^\mu\right)$.

In order to add thermal properties to the above theory one more potential is needed [6]. Therefore a new dimension z^0 has been added. This way the 4-dimensional matter space-time Z is obtained with z^0 play the role of a "matter time". The configuration of the material is now given by four functions:

$$\varsigma^{\alpha} : M \longrightarrow Z$$

α = 0, 1, 2, 3. It has been proved in "KSG" that the following expression for temperature:

$$T = u^{\mu} \frac{\partial \varsigma^0}{\partial x^{\mu}} = u^{\mu} \varsigma^0_{\mu} \tag{1.5}$$

together with the choice of minus the free energy ($f(V,T) = e - TS$) as a Lgrangian of the theory:

$$L = -\sqrt{\bar{g}} \frac{1}{V} f(V,T) = -\sqrt{\bar{g}} \rho f\left(\frac{1}{\rho}, T\right) \tag{1.6}$$

where $\bar{g} = \left|\det g_{\mu\nu}\right|$ and $\rho = \frac{1}{V}$ is the molar matter density.

Finally, the theory of self gravitating fluid has derived by "KSG" from the Lagrangian $L = L_{grav.} + L_{mat.}$, where $L_{grav.}$ is the Einstein-Hilbert Lagrangian:

$$L_{grav.} = \frac{\sqrt{\bar{g}}}{16\pi G} R(\bar{g}) \tag{1.7}$$

R is the 4- dimensional scalar curvature invariant and $L_{mat.}$ is the matter Lgrangian which has been given as before in (1.6)

In Hamiltonian formulation [4], the complete Cauchy data for the theory consist of Cauchy data for both gravitational and hydro-thermo-dynamical field are $\left(P^{kl}, g_{kl}, \varsigma^{\alpha}, p^0_{\alpha}\right)$ [7], where $p^{\mu}_{\alpha} = \frac{\partial L}{\partial \varsigma^{\alpha}_{\mu}}$ are momenta canonically conjugate to hydro-thermo-dynamical potentials ς^{α}. It has been proven that momentum canonically conjugate to ς^0 is equal the entropy current:

$$p^{\mu}_0 = \sqrt{\bar{g}} \rho S u^{\mu} . \tag{1.8}$$

The Hamiltonian giving the time derivatives of Cauchy data in terms of the functional derivatives of the Hamiltonian can be written as [7]:

$$-\delta H = \frac{1}{16\pi G}\int \dot{P}^{kl}\,\delta g_{kl} - \dot{g}_{kl}\,\delta P^{kl} + \dot{p}^{0}_{\alpha}\,\delta\varsigma^{\alpha} - \dot{\varsigma}^{\alpha}\,\delta p^{0}_{\alpha} \qquad (1.9)$$

where dot denotes the time derivative.

Due to the invariance of the theory with respect to space-time deffeomorphisms, we are allowed to impose the following gauge conditions:

$$\varsigma^{\alpha}\left(x^{\mu}\right) = x^{\alpha} \quad . \qquad (1.10)$$

The gauge conditions (1.10) imply also: $\delta\varsigma^{\alpha} = 0$, $\dot{\varsigma}^{k} = 0$ and $\dot{\varsigma}^{0} = 1$. So the Hamiltonian reads:

$$\delta\left(H - \int S\right) = \frac{1}{16\pi G}\int \dot{P}^{kl}\,\delta g_{kl} - \dot{g}_{kl}\,\delta P^{kl} \qquad (1.11)$$

The quantity $\sigma = \int S$ is equal to the total entropy of the system. In the case of spatially-compact space-time, the total gravitational energy vanishes identically $(H = 0)$ and the global Hamiltonian $\mathrm{H} = H - \sigma$ is determined by the total entropy of the system: $\mathrm{H} = -\sigma$ and the quantity S plays the role of minus the Hamiltonian density on the Cauchy surface Σ.

To express the Hamiltonian in terms of canonical parameters $\left(P^{kl}\ ,\ g_{kl}\right)$ we use the constraint equations corresponding to components T^{00}, T^{0}_{k} of the energy momentum tensor as three parameters *X*, *Y* and *Z* defined as follows:

$$\frac{1}{16\pi G}\left(R(g) - \frac{1}{\det g}\left(P^{kl}\,P_{kl} - \tfrac{1}{2}P^{2}\right)\right) = \frac{\rho\mathrm{e} + p n^{2}}{1 - n^{2}} \equiv X\left(g, P\right) \quad (1.12)$$

and

$$Y_{k} \equiv \frac{1}{8\pi G} P^{l}_{k\downarrow l},$$

with its scalar

$$Y \equiv g^{kj}\,Y_{k}\,Y_{j} \equiv Z^{-1}\ \frac{n^{2}\left(\rho\mathrm{e} + p\right)^{2}}{\left(1 - n^{2}\right)^{2}} \qquad (1.13)$$

and

$$Z \equiv \frac{1}{g} \ , \tag{1.14}$$

with

$$n^2 = 1 - \frac{\rho^2}{Z} = 1 - \frac{1}{V^2 Z} . \tag{1.15}$$

From which the following can be obtained

$$\mathrm{e} = V X - \sqrt{Y} \sqrt{V^2 Z - 1} \tag{1.16}$$

$$p(\mathrm{e}, V) = \frac{V Z \sqrt{Y}}{\sqrt{V^2 Z - 1}} - X \tag{1.17}$$

where the pressure $p(\mathrm{e}, V))$ is determined using the state equation.

So the Hamiltonian density (or the entropy density):

$$U \equiv S(\mathrm{e}(X, Y, Z), V(X, Y, Z)) =: U(X, Y, Z) \tag{1.18}$$

and the Hamiltonian evolution equations are

$$\dot{P}^{kl} = 16 \pi G \frac{\delta U}{\delta g_{kl}} \tag{1.19}$$

$$\dot{g}_{kl} = -16 \pi G \frac{\delta U}{\delta P^{kl}} \tag{1.20}$$

Spherically symmetric closed universe

In the case of spherically symmetric compact universe, we have the general form of 3-dimensional metric [8], [9], [10], [11]:

$$d S_{(3)}^2 = e^{\alpha} \ d \chi^2 + e^{\beta} \sin^2 \chi \left(d \theta^2 + \sin^2 \theta \, d \varphi^2 \right) \tag{2.1}$$

Noting that the radial variable $\chi \in [0, \pi]$ and α and β are functions of χ and time $x^0 = t$.

The ADM momenta in this case are [2]:

$$\left.\begin{aligned} P^{11} &= \tfrac{1}{4\pi}\sqrt{g}\, e^{-\alpha}\, a \\ P^{22} &= \tfrac{1}{4\pi}\sqrt{g}\, e^{-\beta}\, b \sin^{-2}\chi \\ P^{33} &= P^{22}\ \sin^{-2}\theta \\ P^{ij} &= 0 \ for\, i \neq j \end{aligned}\right\} \tag{2.2}$$

where a and b are functions of χ and t.

Using (1.11) with the fact that the total gravitational energy vanishes identically $(H = 0)$ in the case of spatially-compact space-time and substituting with equations (2.1) and (2,2) with using (1.12) to (1.17) we get for evolution equations (1.18) and (1.19) will be as defined in (1.18)

$$\left.\begin{aligned} \dot{\underline{A}} &= 16\pi \frac{\delta U}{\delta \alpha} \\ \dot{\underline{B}} &= 16\pi \frac{\delta U}{\delta \beta} \\ \dot{\alpha} &= 16\pi \frac{\delta U}{\delta \underline{A}} \\ \dot{\beta} &= 16\pi \frac{\delta U}{\delta \underline{B}} \end{aligned}\right\} \tag{2.3}$$

with

$$\underline{A} = A\sin^2\chi, \qquad \underline{B} = B\sin^2\chi \tag{2.4}$$

where

$$\begin{aligned} A &= e^{\frac{1}{2}\alpha+\beta}\ a \\ B &= 2e^{\frac{1}{2}\alpha+\beta}\ b \end{aligned} \tag{2.5}$$

Cosmological model with Ultrarelativistic ideal gas

As has been mentioned above, the specification of the equation of state fixes the Hamiltonian of the system (i.e. the entropy function $S = S(\mathrm{e}, V)$). So the dynamical equations (2.3) are well defined.

It is interesting to solve the evolution equations (2.3) in the case when metric functions α and β and momenta functions A and B depend on time only. This corresponds to the homogeneous cosmological model. The most important feature of this problem, in our theory with spherical symmetry, that the vanishing of the space derivatives in equations (2.3) implies that the following relations are always fulfilled the following:

$$\alpha = \beta \tag{3.1}$$

$$2\ A = B. \tag{3.2}$$

The right hand side of equation (2.19) will therefore be:

$$\frac{1}{16\pi}\int \dot{\underline{A}}\,\delta\alpha + \dot{\underline{B}}\,\delta\beta - \dot{\alpha}\,\delta\underline{A} - \dot{\beta}\,\delta\underline{B} \;\; = \frac{3}{16\pi}\int \dot{\underline{A}}\,\delta\alpha - \dot{\alpha}\,\delta\underline{A}\;. \tag{3.3}$$

Performing the integral with respect to χ for both sides of (1,11) with help of (3.3) and (1,18), we get:

$$\delta\left(4\pi\ S\right) = \frac{3}{16\pi}\int \dot{A}\,\delta\alpha - \dot{\alpha}\,\delta A \;\;. \tag{3.4}$$

Accordingly, the system (2.3) will reduce to two equations:

$$\dot{A} = 16\ \pi\ \frac{\delta U}{\delta\ \alpha} \tag{3.5}$$

$$\dot{\alpha} = 16\ \pi\ \frac{\delta U}{\delta\ A} \tag{3.6}$$

(The gravitational constant *G* is taken to be unity) where *U* here is expressed as:

$$U = \frac{4\pi}{3}\ S\,. \tag{3.7}$$

We have therefore a Hamiltonian system with one degree of freedom α and its canonical .momentum *A*. The geometrical parameter *Y* is always zero because it vanishes at the poles and is χ independent. This means that the shift vector vanishes identically. The second feature of this case is that the matter distribution is just a constant. The three geometrical parameters *X, Y* and *Z* are given by:

$$X = \frac{1}{16\pi}\left[6e^{-\alpha} + \tfrac{3}{2}\,e^{-3\alpha}\ A^2\right] \tag{3.8}$$

$$Y = 0 \tag{3.9}$$

$$Z = e^{-3\alpha} \tag{3.10}$$

The molar volume and molar energy are therefore given by:

$$V = \frac{1}{\sqrt{Z}} \tag{3.11}$$

$$\mathrm{e} = \frac{X}{\sqrt{Z}} \tag{3.12}$$

and the Hamiltonian is given by:

$$U = \frac{4\pi}{3} F(X, 0, Z) = \frac{4\pi}{3} S\left(\frac{X}{\sqrt{Z}}, \frac{1}{\sqrt{Z}}\right) \tag{3.13}$$

Suppose that we have the entropy of a monatomic ideal gas which described by:

$$S = R \ \ln\left((\mathrm{e} - m)^{3/2} V\right) \tag{3.14}$$

where m is rest mass per mole and R is the gas constant. We restrict ourselves to the ultrarelativistic ideal gas where $m = 0$, as:

$$S = R \ \ln\left(\mathrm{e}^{3/2} V\right) \tag{3.15}$$

The dynamics in this case is given by the following Hamiltonian:

$$S = F(X, 0, Z) = \frac{1}{4} R \ \ln\left\{\frac{X^6}{Z^5}\right\} = \frac{3}{4} R \ \ln\left\{e^{\alpha}\left(6e^{\alpha} + \tfrac{3}{2} e^{-\alpha} A^2\right)\right\} \tag{3.16}$$

Therefore, equations (3.5) and (3.6) become:

$$\dot{A} = (4\pi)^2 R \left[\frac{6e^{-\alpha} - \frac{1}{2} e^{-3\alpha} A^2}{2e^{-\alpha} - \frac{1}{2} e^{-3\alpha} A^2}\right] \tag{3.17}$$

$$\dot{\alpha} = -(4\pi)^2 R \left[\frac{2e^{-3\alpha} A}{2e^{-\alpha} - \frac{1}{2} e^{-3\alpha} A^2}\right] \tag{3.18}$$

Dividing (3.17) by (3.18) we get

$$A\dot{A} = -(3 - \tfrac{1}{4} e^{-2\alpha} A^2)\, \dot{\alpha} e^{2\alpha}. \tag{3.19}$$

Performing the following substitution:

$$A = a\, e^{\alpha}\,. \qquad (3.20)$$

In (6.19) we get:

$$a\,\dot{a} = -3\,\dot{\alpha}\,(1 + \tfrac{1}{4}\,a^2)\,. \qquad (3.21)$$

From which we get:

$$\frac{d}{dt}\left[\ln(1 + \frac{1}{4}a^2) + \frac{3}{2}\alpha\right] = 0\,. \qquad (3.22)$$

Which implies that:

$$a^2 = 4\,(e^{(C - \frac{3}{2}\alpha)} - 1) \qquad (3.23)$$

where C is a constant of motion. Substituting by (3.23) in (3.17), we get:

$$\dot{\alpha} = 2\,R\,\frac{(4\pi)^2}{e^C}\,e^{\alpha/2}\,\sqrt{e^{(C - \frac{3}{2}\alpha)} - 1}\,. \qquad (3.24)$$

If we make the following substitution and then integrating:

$$\phi = e^{(C/3\, -\, \alpha/2)}\,, \qquad (3.25)$$

we get:

$$\frac{1}{(3)^{1/4}}F\left[arc\,\cos\left(\frac{\phi - 1 - \sqrt{3}}{\phi - 1 + \sqrt{3}}\right),\, \sin 15^{o}\right] = C_o\,(t_{max} - t) \qquad (3.26)$$

where $C_0 = R\left[\dfrac{4\pi}{e^{C/3}}\right]^2$, $t_{\max}$ is the time for which ϕ tends to infinity and F the elliptic integral of the first kind. This solution contains two arbitrary constants C and $t_{\max}$. From (3.26) we get inversely:

$$e^{\alpha} = \left[\frac{e^{C/3}\,(1 - Cn(\underline{t}))}{(1 + \sqrt{3}) - (1 - \sqrt{3})\,Cn(\underline{t})}\right]^2 \qquad (3.27)$$

where $Cn(\underline{t})$ is Jacobean elliptic function and $\underline{t} = C_o(t_{\max} - t)$. With some analysis we can see that the function e^{α} has minimum at $t_{\max} = t$ and the value of the function at this time is:

$$e^{\alpha} \xrightarrow[t_{\max} = t]{} 0, \qquad (3.28)$$

and have maximum when:

$$C_o(t_{\max} - t) = 2\mathrm{K}, \qquad (3.29)$$

where K is a complete elliptic integral of the First kind (can be obtained from mathematical tables) and the value of' the function at this time is:

$$e^{\alpha} = e^{2C/3} \qquad (3.30)$$

From (3.20) and (3.23) we can have the solution for A as:

$$A^2 = 4[e^C e^{\alpha/2} - e^{2\alpha}], \qquad (3.31)$$

which implies that the inequality:

$$C \geq \frac{3}{2}\alpha \qquad (3.32)$$

must, be fulfilled always which is true for the maximum value of e^{α} given by (3.29). It is clear from this discussion that the collapse occurs in finite time.

The expressions for physical parameter's V as molar volume, e as the molar energy, p the pressure, and T the temperature are derived in the following form:

$$V = \frac{1}{\rho} = \frac{1}{\sqrt{Z}} = \left[\frac{e^{C/3}\,(1 - Cn(\underline{t}))}{(1+\sqrt{3}) - (1-\sqrt{3})\,Cn(\underline{t})}\right]^3 \qquad (3.33)$$

$$\mathrm{e} = \frac{X}{\sqrt{Z}} = \left[\frac{6\,e^C}{16\,\pi}\right]\left[\frac{(1+\sqrt{3}) - (1-\sqrt{3})\,Cn(\underline{t})}{e^{C/3}\,(1 - Cn(\underline{t}))}\right]^2 \qquad (3.34)$$

$$p = \tfrac{2}{3}X = \left[\frac{e^C}{4\pi}\right]\left[\frac{(1+\sqrt{3}) - (1-\sqrt{3})\,Cn(\underline{t})}{e^{C/3}\,(1 - Cn(\underline{t}))}\right]^5 \qquad (3.35)$$

$$T = \tfrac{4}{9}\,\frac{1}{R}\,\frac{X}{\sqrt{Z}} = \left[\frac{e^C}{6\,\pi\,R}\right]\left[\frac{(1+\sqrt{3}) - (1-\sqrt{3})\,Cn(\underline{t})}{e^{C/3}\,(1 - Cn(\underline{t}))}\right]^2 \qquad (3.36)$$

In this model we note that parameter e^{α} is always decreasing with time as the space collapse. This supports the conjecture of recontraction of the closed universe [12]. The parameter A increases with time until its evolution reversed and begin to decrease. The pressure and temperature are increasing continually as collapse is going on.

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