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HALLEY'S FUNCTION FOR REAL POLYNOMIALS IS INCREASING

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Abstract. In this paper we want to show that Halley's function for real polynomial is an increasing rational homeomorphism map on $\mathbb{R}$.

Keywords: Halley's function, Derivative of Halley's method, homeomorphism, Increasing Function.

Introduction

Halley's method is an elegant method for finding roots and a third-order algorithm. Such an algorithm converges cubically insofar as the number of significant digits eventually triples with each iteration. And not only does the first derivative of a third-order iteration vanish at a fixed point, but so does the second derivative. In this paper, we recall some definitions, theorems for Halley's function for a real polynomial and the derivative of Halley's function. Then we conclude that Halley's function for real polynomial is an increasing rational homeomorphism map on $\mathbb{R}$.

0.1 Halley's method for real polynomials

In this section, our objective is to study the iteration of Halley's function associated with a polynomial p of degree d with real coefficients and only real (and simple) zeros x_k, $1 \leq k \leq d$. This method is equivalent to iterating the rational map

$$H_p(z) = z - \frac{2p'(z)p(z)}{2(p'(z))^2 - p(z)p''(z)}, \tag{0.1.1}$$

where

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{d-1} z^{d-1} + a_d z^d.$$

So if $p(z)$ has degree d and has distinct roots, then by a simple calculation $H_p(z)$ is a rational map of degree $2d - 1$. As for the case of Newton's method, the roots of $p(z)$ are fixed points of $H_p(z)$, although other fixed points exist as well. Since we are assuming that the roots of $p(z)$ are distinct, the critical points of $p(z)$ are also fixed points under Halley's method.

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0.2 Derivative of Halley's method

The derivative of Halley's method is

$$H_p'(z) = -\frac{(p(z))^2 S[p](z)}{2\left(p'(z) - \frac{p(z)p''(z)}{2p'(z)}\right)^2}, \tag{0.2.1}$$

where $S[p](z)$ is the Schwarzian derivative of $p(z)$, that is

$$S[p](z) = \frac{p'''(z)}{p'(z)} - \frac{3}{2}\left(\frac{p''(z)}{p'(z)}\right)^2. \tag{0.2.2}$$

From expression (0.2.1), we can see that the roots are super-attracting fixed points, but of one degree higher order than for Newton's method.
As we know that the degree of Halley's function is $2d-1$, where d is the degree of the polynomial p, there are $4d-4$ critical points, $2d$ of them coincide with the roots x_k, and $2d-4$ are free critical points placed at points where the Schwarzian derivative of $p(z)$ vanishes.

Remark 0.2.1. The second derivative of H_p vanishes at x_k, where as the second derivative of N_p does not, the graph of H_p is flatter than that of N_p near the fixed point. This accounts for the difference in speed at which the two algorithms converge (see [6], [3] for details). In general, the higher the order, the flatter the graph, the faster convergence.

Theorem 0.2.1. *Let*

$$H_p(z) = z - \frac{2p'(z)p(z)}{2(p'(z))^2 - p(z)p''(z)},$$

where p is a polynomial with real (and simply) distinct zeros. Then H_p has no real pole.

Proof. We will show that

$$(p')^2 - pp'' > 0 \quad on \ \mathbb{R},$$

which is known as Polya's result.
Write

$$(p')^2 - pp'' = p^2\frac{(p')^2}{p^2} - p^2\frac{pp''}{p^2} = p^2\left(\left(\frac{p'}{p}\right)^2 - \frac{p''}{p}\right).$$

We know that

$$\left(\frac{p'}{p}\right)^2 = \left(\sum_{j=1}^{d}\frac{1}{z-x_j}\right)^2,$$

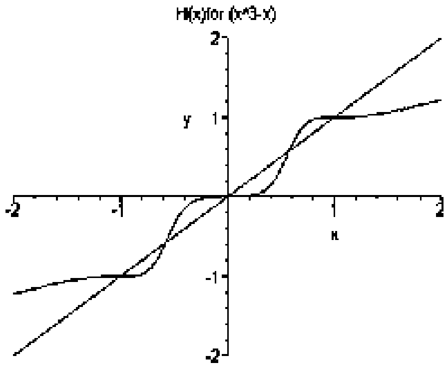

Figure 1: Halley's function for the polynomial $p(x) = x^3 - x$.

where x_j are roots of p, $1 \le j \le d$, hence

$$\frac{p''}{p} = \left(\sum_{j=1}^{d} \frac{1}{z - x_j}\right)^2 - \sum_{j=1}^{d} \frac{1}{(z - x_j)^2}.$$

From

$$\sum_{j=1}^{d} \frac{1}{(z - x_j)^2} > 0, \qquad z \in \mathbb{R},$$

it follows that

$$\left(\frac{p'}{p}\right)^2 > \frac{p''}{p},$$

hence

$$2(p')^2 - pp'' > 0.$$

Thus H_p does not have any real pole.

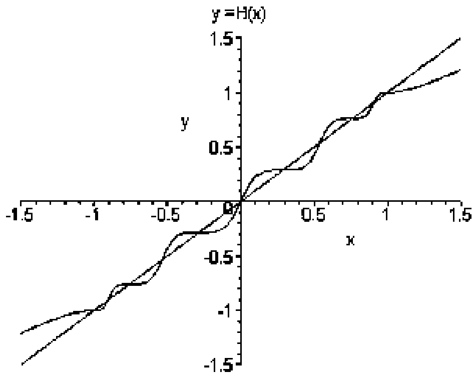

Figure 2: Halley's function for the polynomial $p(x) = x^6 - \frac{5}{3}x^4 + \frac{5}{7}x^2 - \frac{1}{21}$.

Theorem 0.2.2. *Let $H_p(z)$ be a Halley's function for a polynomial $p(z)$, then $H_p'(z) \geq 0$ on $\mathbb{R}$.*

Proof. We know that

$$H_p'(z) = -\frac{(p(z))^2 S[p](z)}{2\left(p'(z) - \frac{p(z)p''(z)}{2p'(z)}\right)^2},$$

where $S[p](z)$ is the Schwarzian derivative of $p(z)$, that is

$$S[p](z) = \frac{p'''(z)}{p'(z)} - \frac{3}{2}\left(\frac{p''(z)}{p'(z)}\right)^2 = \frac{2p'p''' - 3(p'')^2}{2(p')^2}.$$

To show that $H_p'(z) \geq 0$, we have to prove that $S[p](z) < 0$. By the same proof as before, we can see that $(p'')^2 - p'p''' > 0$, then $2p'p''' - 3(p'')^2 < 0$. Thus $S[p](z) < 0$, implies $H_p'(z) \geq 0$. □

***Conclusion* 1.** From theorems (0.2.1) and (0.2.2), we conclude that H_p is an increasing rational homeomorphism map on $\mathbb{R}$.

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