Distributive Property Overgeneralizations in Mathematical Functions

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Abstract:

Researcher argued that there are tight links between learners' awareness of binary operations properties' and their beliefs about functions. This paper went deeply into analyzing the of that relationship through reviewing some wrong types of students' beliefs about functions in view of distributive binary operations in mathematics curricula. This paper showed the sensibility of some wrong types of students' beliefs about functions and how these wrong types of students' beliefs about functions assimilate overgeneralizations for distributive properties of binary operations.

Keywords: Functions; Overgeneralization; Generalization; Distributive Property; Mathematical operations; algebraic properties; Constant errors; transmission of the learning; linear function; logarithm function; trigonometric function; Binary operations.

Introduction:

One of important teaching mathematics' aims is to create and develop educational situations and mathematical activities in order to activate students concluding generalizations and applying these generalizations in all appropriate cases and to use them in solving mathematical problems.

Mathematical generalization does not apply to a single mathematical example, but applies to a wide range of mathematical examples and attitudes. The generalization is an abstract of mathematical relationships involving such examples and mathematical attitudes.

The familiarity of the learner with the realization of a mathematical relationship or idea, in more than one position, leads to easy access to generalization. But also, this familiarity can lead to overgeneralizations of the idea on similar positions.

Some of the literature, which dealt with the overgeneralizations, indicated that large number of situations in which the learner is familiar with an idea, leads to the occurrence of overgeneralizations of that idea on other positions similar to those where the learner achieved the idea(Abu Allam, 1992; Atia, 1990; El- Tawab, 1994).

Previous studies, as (Leinhardt, Zaslavsky, & Stein, 1990; Tirosh, 1991; Zaslavsky & Peled, 1996) attributed some of the difficulties encountered by the learners in their study of mathematics at higher educational levels, to the overgeneralizations of their early experiences with the arithmetic operations they studied in the primary stage.

Distributive Property:

For any binary operation(θ), we say that operation (θ) is distributive over other binary operation (+) if, for any three elements (x, y, and z of S), this operation satisfies:

- $x \theta (y + z) = (x \theta y) + (x \theta z)$, (left – distributive)
- $(y + z) \theta x = (y \theta x) + (z \theta x)$. (right – distributive)

Thus, if a binary operation θ is distributive over a binary operation (+), then operation (+) satisfies left– and right- distributive over operation (+), and vice versa.

Few of binary operations, are right– distributive over another binary operation, and they are not left- distributive over it. At the other hand, few binary operations that are left– distributive over another binary operation, and they are not right- distributive over it.

If a binary operation (θ) is left – distributive over a binary operation (+), and (θ) is not Commutative, it would not be right-distributive. In addition, if a binary operation (θ) is right–distributive over a binary operation (+), and (θ) is not Commutative, it would not be left - distributive.

For example, division operation is right– distributive over (addition & subtraction), division is not Commutative, and division is not left - distributive over (addition & subtraction).

In some structures, we can find two binary operations, mutually related to each other by the distributive property; for example, in Boolean algebra the binary operations, " ∧ " meet, "∨" join, are mutually related to each other by the distributive property.

learner's experiences in distributive binary operations:

There are many situations, may hundreds or thousands, learner pass through in which learner may do one or more of the following activities: receiving information about distributive binary operation, generalizing, practicing or solving problems using distributive binary operations.

The study of distributive property starts in primary school, and then it continue in all scholar life. The learner passes through several situations, in which he sees that distributive property is so frequently. These situations spread over all stages as primary, senior primary and secondary schools. These situations continue through studying mathematics in the university. Through moving from grade to the next, and from stage to upper, the learner find the binary operations' distributivity seems more reasonable and necessary to do hard activities in short time and in brief with comfortable manner.

From year to next, learners apply more implications of distributive property for binary operations. These implications increase as learners grow up from year to next. The learner passes through increasing situations in which he sees that distributive operation so fantastic and useful to do much of calculations easily. The following is a brief overview of the experiences of the distribution property that students acquire in general education.

Distributive binary operations in primary schools' mathematics

Primary schools introduce situations about Distributive binary operations, as:

= Multiplication distributivity left– and right- over addition in each of (natural numbers, positive fractions, positive decimals).

= Multiplication distributivity left– and right- over subtraction in each of (natural numbers, positive fractions, positive decimals).

= division distributivity right- over addition in each (natural numbers, positive fractions, positive decimals).

= division distributivity right- over subtraction in each (natural numbers, positive fractions, positive decimals).

Thus primary students practice distributive property at least tens of times in each one of the four subjects.

Distributive binary operations in senior primary schools' mathematics:

Senior primary schools introduce several situations about distributive binary operations, as:

= Applying same specific distributive operation operations those taught in primary school, and the following,

= Multiplication distributivity left– and right- over addition & subtraction in each of numbers, i. e., decimal fractions, fractions, negative integers, positive integers, integers, rational numbers, real number and irrational numbers.

= Division distributivity right- over subtraction in each of numbers, i. e., decimal fractions, fractions, negative integers, positive integers, integers, rational numbers, real number, irrational numbers).

= The intersection distributivity left– and right- over the union operation in set theory; $A\cap (B\cup C) = (A\cap B) \cup (A\cap C)$, $(B\cup C) \cap A = (B\cap A) \cup (C\cap A)$; for sets A, B and C.

= The union distributivity left– and right- over the Intersection operation in set theory, $A\cup(B\cap C) = (A\cup B) \cap (A\cup C)$, $(B\cap C) \cup A = (B\cup A) \cap (C\cup A)$; for sets A, B and C.

= Multiplication distributivity left– and right- over (addition &subtraction) in dealing with some equations, i. e., linear, quadratic, the three basic trigonometric, absolute value and Exponential equations, e. g., in the absolute value equations; If: $y = 2x$, $4y = 22x$, $16y = 23x$, $y\{ 4y \pm 16y \} = y \times \{ 16y + 4y \} = \{2x \times 22x \}+ \{2x \times 23x\}= 23x + 24x$

= Division distributivity right- over (addition & subtraction) in dealing with some equations, e.g., linear, quadratic, the three basic trigonometric, absolute value and exponential equations.

= Multiplication distributivity left - over (addition &subtraction) in dealing with some functions as linear function, quadratic function, three basic trigonometric function, absolute value function and exponential function, e.g., If: y_1, y_2, y_3 are linear functions, then: $y_1 \times \{ y_2 \pm y_3 \} = \{ y_1 \times y_2 \} \pm \{ y_1 \times y_3 \}$.

= Multiplication distributivity right- over (addition &subtraction) in dealing with some functions as linear function, quadratic function, three basic trigonometric function, absolute value function and exponential function, e.g., in trigonometric functions; If: $y_1 = \mathit{sin}$ x, $y_2 = \mathit{cos}$ x, $y_3 = \mathit{tan}$ x, $\{ y_2 \pm y3\} \times y_1 = \{ \mathit{cos}\ x \pm \mathit{tan}\ x\} \times y_1 = \{ \mathit{cos}\ x \times \mathit{sin}\ x \} \pm \{ \mathit{tan}\ x \times \mathit{sin}\ x \}$.

= Division distributivity right- over (addition &subtraction) in dealing with some functions as linear function, quadratic function, three basic trigonometric function, absolute value function and exponential function, e. g., If: y_1, y_2, y_3 are exponential functions, then $\{ y_2 \pm y_3\} \div y_1 = \{ y_2 \div y_1 \} \pm \{ y_3 \div y_1\}$.

These kinds of situations about Distributivity binary operations in senior primary school show how the learner applies division property in encountered situations.

Distributive binary operations in Secondary schools' mathematics:

Secondary schools introduce more situations about Distributive binary operations, as:

= Applying specific distributive situations those taught in primary & senior primary school, and the following,

= Multiplication distributive left– and right- over addition in complex numbers.

= Multiplication distributivity right- over subtraction in complex numbers.

= Division distributivity right- over addition in complex numbers.

= Division distributivity right- over subtraction in complex numbers.

= Dot Product satisfies left– and right- distribution over addition & subtraction of Vectors (two-dimensional space, three-dimensional space… n dimensional space); A. (B+C) = (A.B) + (A.C); where A, B and C are vectors.

= cross product satisfies left– and right- distribution over addition & subtraction of Vectors (two-dimensional space, three-dimensional space… n dimensional space); $A \times (B+C) = (A \times B) + (A \times C)$; where A, B and C are vectors.

= Scalar multiplication is right and left- distributive over addition of vectors (two-dimensional space, three-dimensional space… n dimensional space), $k(A+B) = kA + kB$; where A, B are vectors; k is real number.

= Scalar Multiplication of Matrices is right and left- distributive over addition of matrices, c (A+B) = (cA) + (cB), (A+ B) c = (Ac) + (Bc), where c is real number, matrices A & B of m-by-n.

= Scalar Multiplication of Matrices is right and left- distributive over addition and subtraction of real numbers, (c ± d) A = (cA) ± (dA), A (c ± d) = (Ac) ± (Ad), where A is matrix & d, c; d are real numbers.

= Multiplication is left- distributive over addition of matrices, A × (B + C) = (A×B) + (A×C), where A is m-by-n matrix, and B, C are n-by-k matrices.

= Multiplication is right- distributive over addition of matrices, $(B + C) \times A = (B\times A) + (C\times A)$, where: B, C are m-by-n matrices, A is n-by-k matrix

= Multiplication is left- distributive over addition & subtraction for determinants of matrix; $\det(A) \times \{\det(B) \pm \det(C)\} = \{\det(A) \times \det(B)\} \pm \{\det(A) \times \det(C)\}$, for matrices A, B and C.

= Multiplication is right- distributive over addition & subtraction for determinants of matrix; $\{\det(B) \pm \det(C)\} \det(A) = \{\det(B) \times \det(A)\} \pm \{\det(C) \times \det(A)\}$, for matrices A, B and C.

= Multiplication distributivity right- over (addition &subtraction) in dealing with some equations, e.g. cubic, Logarithms, polynomial, basic trigonometric, trigonometric polynomials, matrix polynomials, and exponential equations.

= Multiplication distributivity left - over (addition &subtraction) in dealing with some functions as cubic functions, polynomial functions, square root function, complex functions, exponential function, logarithm function, trigonometric functions, matrix functions, exponential polynomial, inverse functions, matrix polynomials, power function, rational functions and trigonometric functions, e.g., If: z_1, z_2, z_3 are complex functions, then $z_1 \{ z_2 \pm z_3 \} = \{ z_1 \times z_2 \} \pm \{ z_1 \times z_3 \}$.

= Multiplication distributivity right- over (addition &subtraction) in dealing with some functions as linear cubic functions, polynomial functions, square root function, complex functions, exponential function, logarithm function, trigonometric functions, matrix functions, exponential polynomial, inverse functions, matrix polynomials, power function, rational functions and trigonometric functions, e.g., If: z1, z_2, z_3 are complex functions, then $\{ z_2 \pm z_3\}\times z_1 = \{ y_2 \times z1\}\pm \{ z_3 \times z_1\}$

= Division distributivity right- over (addition &subtraction) in dealing with some functions as cubic functions, polynomial functions, square root function, complex functions, exponential function, logarithm function, trigonometric functions, matrix functions, exponential polynomial, inverse functions, matrix polynomials, power function, rational functions and trigonometric functions, e. g., If: y_1, y_2, y_3 are power functions, then $\{ y_2 \pm y3\}\div y_1 = \{ y_2 \div y1 \}\pm \{ y_3 \div y_1\}$.

These situations about division binary operations in secondary school indicate that learner apply distributive in so many binary operations, and this applying distributive property becomes more familiar as the learner moves from grade to upper and from educational stage to the next.

Overgeneralizations and random errors

We have to distinguish between overgeneralizations and random errors. If a correct generalization of a definite mathematical idea is used in an inappropriate situation, this case represents overgeneralization. When the correct generalization of that idea is used in appropriate situation, with mistakes in, that is random errors not overgeneralization. For example; in solving each of: $(6x^4)C/(2x^4 + 3x^2 + 6)$, $(2x^4 + 3x^2 + 6)/(6x^2\ )$ as following:

1- $\frac{6x^4}{(2x^4+3x^2+6)} = \frac{6x^4}{2x^4} + \frac{6x^4}{3x^2} + \frac{6x^4}{6} = 3 + 2x^2 + x^4$

2- $\frac{(2x^4+3x^2+6)}{6x^2} = \frac{2x^4}{6x^2} + \frac{3x^2}{6x^2} + \frac{6}{6x^2} = 2x^2 + \frac{1}{2} + \frac{6}{x^2}$

The two solutions above involve division distributive over addition in rational functions; the first one involves division distributive left– over an addition, whereas the second involves division distributive right–over an addition. The second is appropriate for applying right distributivity, it is applied in correct way, but it includes errors can be pointed as random mistakes not overgeneralization. The first one is inappropriate for using left distributivity, which is an overgeneralization of distributive property.

Overgeneralizations & Impact transmission of the learning:

The learner passes through many educational situations, and detects some generalizations of his experiences with some of these situations. Some of the ideas and principles learned previously, the individual Benefits them in other new situations. This represents an impact transmission of the learning.

Abd-Elkader (Abd-Elkader, 1998) Stated that the generalization is the core of any process of impact transmission of the learning. impact transmission of the learning happens whenever the individual can apply experience gained on new situation (Atia, 1990, p. 94).

Impact transmission of the learning is that training in any area or activity spill over to activities in other situations comparable with the previous learned. The impact of the previous training appears later in events or situations similar to which trained on(El mamomary, 2011).

The Literature review (Abd-Elkader, 1998; Abu Allam, 1992; Atia, 1990; El- Tawab, 1994), shows three types of impact transmission of the learning, as following**:**

1- Positive impact transmission: The positive impact transmission of the learning occurs when the learner's previous experiences contribute to learning new attitudes. The generalizations reached by the learner, in the positive impact transmission, are correct.

There are many mathematical situations in which a positive impact transmission occurs in mathematics learning. For example, students pass through many mathematical situations in which they see the applicability of distribution to addition and multiplication' properties among numbers. The student may benefit from this when studying distribution for the union and the intersection of two sets.

2- Negative impact transmission: The negative impact transmission of the learning occurs when the learner's previous experiences hinders learning new attitudes. Students in this case suffer from the overgeneralization of some ideas they have already learned, and consider it applicable to some new situations.

3- Zero impact transmission: zero impact transmission of the learning means that the learner's previous experiences of certain task does not affect negatively or positively on the learning of a new task.

(Atia, 1990)Pointed that impact transmission of the learning from one situation to another, occurs as far as there is between the two positions of identical elements or components. If the similarity between these elements increased, the impact transmission of the learning is increased.

(Abu Allam, 1992) Argue that there is a similarity between the positive and the negative impact transmission of the learning. This similarity lies in the fact that both of these types of impact transmission include the application of generalizations and ideas that had learned in certain educational situations to new situations and problems. The two types of impact transmission of the learning differ from each other in the accuracy of the resulting generalization.

In view of above:

1. Learners' with two similar situations in one or more elements may lead to an impact transmission of the learning from one situation to the other. The probability of this impact transmission increases with the similarities between the two situations.
2. The positive impact transmission of the learning leads to get correct generalizations of some ideas studied in previous situations. This contributes to the learning of subsequent situations and supports some ideas in new learning similar positions.
3. The negative impact transmission of the learning leads to overgeneralizations of some ideas studied in previous situations. This hinders the learning of some ideas in new later situations.

It is clear from the relationship between the generalizations reached by the learner and the impact transmission of the learning, that overgeneralizations are a product of an impact transmission of the learning.

One, who peruses some constant errors in mathematics among students, realizes that much of these errors results from overgeneralizations of the four basic arithmetic operations' properties. Distributivity overgeneralization can be a look of negative impact transmission of learning distributive property.

(Zaslavsky & Peled, 1996) pointed to factors contribute difficulties encountered by mathematics teachers and student teachers associated with the concept of binary operation;

findings suggest two main inhibiting factors: one related to the overgeneralization of the properties of basic binary operations and the other related to pseudo-similarities attributed to these properties.

Distribution overgeneralizations types in functions:

In handling with functions, the literature shows constant error types can be referred as over generalizing of distributive properties. These errors can be any of tow types as following;

1. Overgeneralization between functions: this type arises in doing binary operations between two or more functions. This type is clear as literally application of distributive property of distributive binary operation. This type can be expressed as the following: f(c) ÷ { f(a) + f(b)} = { f(c) ÷ f(a)} + { f(c) ÷ f(b)}, f(c) ÷ { f(a) - f(b)} = { f(c) ÷ f(a)} - { f(c) ÷ f(b)}. These errors represent overgeneralizations of division distributive.
2. Overgeneralizations inside function: This type is compromise dealing. When analyzing this type; it is apparent that it includes double contrary treatments;
   a) Handling the function's symbol as it is means. Then the action is correct. For example, if $y = 3x^3 + 1$; then $f(3) = 3(3)^3 + 1 = 81 + 1 = 82$.
   b) Handling the function's symbol as it is variable or real number, then applying the operation's properties of four basic operations. For example, the following errors:

$(x + y)^n = x^n + y^n$ $\sqrt[n]{(x^n + y^n)} = x + y$

$(x - y)^n = x^n - y^n$ $\sqrt[n]{(x^n - y^n)} = x - y$

$a^{m+n} = a^m + a^n$

$a^{m-n} = a^m - a^n$

See literature (Davis & McKnight, 1979; Laursen, 1978; Mansoor, 2002; Matz, 1980; Schwartzman, 1977).

these above errors are “an overgeneralization of the property f (a + b) = f (a) + f (b), which applies only when f is a linear function, to the form f (a * b) = f (a) * f (b), where f is any function and * any operation” (Olivier, 1989, 10-19, p. 10).

These errors above include applying distribution property while it is inappropriate. They can be classed as Distribution Overgeneralization Inside Function (DOIF).

Patterns of distribution overgeneralization inside functions:

constant errors are common among learners in handling with functions. Many of these errors can be classified as over generalizing distributive properties. These errors happen in much mathematics subjects as in functions.

A survey of the literature shows that DOIF can be classed into eight patterns, as following:

- Exponent distributivity over addition and subtraction of the Power functions’ base.
- Base distributivity over addition, subtraction and division of the Exponential functions’ exponent.
- Radical sign distributivity over each of addition and subtraction of radicands in root function.
- Logarithm symbol distributivity over four basic operations, in logarithm function.
- Trigonometric function symbol distributivity over addition and subtraction.
- polynomial function symbol distributivity over four basic operations.
- Division Distributivity by rational functions.

- Absolute value function symbol distributivity over addition and subtraction.

The following, is explication for these DOIF errors' patterns.

Pattern 1: exponent distributivity over addition and subtraction of the Power functions' base

Power functions: $y = a\,x^{b}$, where a, and b are constants, real numbers, $b \neq 0$

When a = 1, $y = x^{b}$

When a = 1, power function became: $y = x^{b}$

When substituting with values as (x= c±d) to the base of power function xb, this causes difficulties to some students.

A survey of the literature (Egodawatte & Stoilescu, 2015; Fischbein, 1994; Fischbein & Barash, 1993; Kirshner, 1985; Laursen, 1978; Mansoor, 2002; Matz, 1980; Olivier, 1989; Schwartzman, 1977) shows those senior primary and even secondary schools' students have errors in substituting with values to the base of power function as:

$3(a+b)^2 = 3a^2 + 3b^2$

$(a+b)^5 = a^5 + b^5$

$(x+y)^n = x^n + y^n$

$(x-y)^n = x^n - y^n$

$(6+b)^2 = 36 + b^2$

$(x+y)^5 = x^5 + y^5$

These errors can be seen as examples of overgeneralizing distributivity. Kirshner concluded that advanced students tend to go through a phase of overgeneralizing before achieving fluency in manipulative skill(Kirshner, 1985)

The two statements ($(6+b)^2 = 36 + b^2$,$(x+y)^5 = x^5 + y^5$) include substituting the base with values: (6 + b and x + y) in order. Analyzing these errors shows similarity between the errors, as assimilated in table 1.

Table 1. Analysis of substituting with values to the base of power function

dealing	$(6+b)^2 = 36 + b^2$	$(x+y)^5 = x^5 + y^5$
similar	**(a+b)c = ac + bc**	**(a+b)c = ac + bc**
Verbal description	**Distributive of the exponent over addition of the base**	
analysis	**Distributivity overgeneralization**	

Table 1 shows that, the two wrong statements: are including "exponent distributivity over addition and subtraction of the base". These errors are examples of DOIF in dealing with substituting with values to the base of power function.

Pattern 2: base distributivity over addition, subtraction and division of the Exponential functions' exponent

Exponential function: $y = a\,b^x$, where a, and b are constants, real numbers, $a \neq 0$

When a = 1, exponential function $y = b^x$

Substituting with values as ($x = c \pm d$) to the exponent of exponential function bx, causes difficulties to some students.

Literature review (Kirshner, 1985; Mansoor, 2002; Matz, 1980; Schwartzman, 1977) shows that senior primary and even secondary schools' students have errors in substituting with values to the exponent of exponential function as:

$(17)^{3x+5} = (17)^{3x} + (17)^5$

$(6)^{3\div 5} = (6)^3 \div (6)^5$

$(3)^{2-x} = (3)^2 - (3)^x$

$a^{m+n} = a^m + a^n$

The four statements include substituting the exponents with values: (m +n, 3x+5, 3÷5 and 2-x) in order. Analyzing these errors shows similarity between them. Two of the three statements above, are assimilated in table 2.

Table 2. Analysis of substituting with values to the exponent of exponential function

Error	No. 1	No. 2
dealing	$(17)^{3x+5} = (17)^{3x} + (17)^{5}$	$(6)^{3\div 5} = (6)^{3} \div (6)^{5}$
similar	a(b+c) = (ab)+(ac)	a(b ÷c) = ab ÷ ac
Verbal description	Distributive of the base over addition of the exponent.	Distributive of the base over division of the exponent
analysis	Distributivity overgeneralization	

Table 2 shows that, the two wrong statements: are including "base distributivity over each of addition, subtraction and division of the exponent". These errors are examples of DOIF in dealing with substituting with values to the exponent of exponential function.

Pattern 3: radical sign distributivity over each of addition and subtraction of radicands in root function

(Matz, 1980) indicated that following error is an example for One of the largest and most frequently occurring class of errors in the high school; $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

(Kirshner, 1985) indicated that each of the following error is common for beginning algebraists:

$$\sqrt[n]{(x^n + y^n)} = x + y$$

$$\sqrt[n]{(x^n - y^n)} = x - y$$

Also (Mansoor, 2002) showed similar results among senior primary and secondary schools students, as:

$$\sqrt{6^2 + 3^2} = 6 + 3$$

$$\sqrt[3]{(3^3 + 4^3 + 5^3)} = 3 + 4 + 5$$

Why these errors happen?

The following Table 3 Indicates radical sign distributing over each addition and subtraction of radicands in root function.

Table 3. Distribution overgeneralization in dealing with root function as one of DOIF patterns

solution	similar	Analysis
$\sqrt{6^2+3^2}$		Beginning
$\sqrt{6^2+3^2} = \sqrt{6^2}+\sqrt{3^2}$	a(b + c) = ab + ac	Distributivity overgeneralization
$\sqrt{6^2}+\sqrt{3^2} = (3^2)^{\frac{1}{2}} + (6^2)^{\frac{1}{2}}$	$\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$	
$(6^2)^{\frac{1}{2}} + (3^2)^{\frac{1}{2}} = 6 + 3$	$(x^m)^n = x^{mn}$	

From Table 3, it is apparent, that the step which is: $\sqrt{6^2+3^2} = \sqrt{6^2}+\sqrt{3^2}$, means that some students distribute “radical” symbol over addition and subtraction of radicands when dealing with root function. This is similar to distributing multiplication over addition and subtraction in numbers.

These errors in applying distribution property over addition and subtraction in dealing with root function represent DOIF.

Pattern 4: logarithm symbol distributivity over four basic operations, in logarithm function

Literature survey (Schwartzman, 1977; Laursen, 1978; Davis, 1979; Matz, 1980; Olivier, 198; Kirshner, 1985; Fischbein, 1993; Fischbein, 1994; Mansoor, 2002; Egodawatte, 2015) shows that secondary schools’ students have errors in substituting with values to the logarithm function as:

1. *log* (*a*+*b*) = *log a* + *log b*
2. *log* (x/y) = *log* x ÷ *log* y,
3. *log* (a+b+c) = *log*(a) + *log*(b) + *log*(c)

The last wrong statements contain similar procedures. They are assimilating in table 4.

The following Table 4. includes analysis of the statement; *log* (x/y) = *log* x ÷ *log* y, *log* (a+b+c) = *log*(a) + *log*(b) + *log*(c)

Table 4. analysis of substituting errors in logarithm functions.

dealing	*log* (x/y) = *log* x ÷ *log* y	*log* (a+b+c) = *log*(a) + *log*(b) + *log*(c)
similar	a(b÷c) = (ab)÷(ac)	a(b + c + d) = ab + ac + ad
Verbal description	Distributive of function symbol over division.	Distributive of logarithm symbol over addition.
Symbolic description	f(a÷b) = f(a) ÷ f(b)	f(a+b+c) = f(a) + f(b) +f(c)
analysis	Distributivity overgeneralization	Distributivity overgeneralization

Table 4 shows that, the two wrong statements: *log* (x/y) = *log* x ÷ *log* y, *log* (a+b+c) = *log* (a) + *log* (b) + *log*(c) are including "logarithm function symbol distributivity over addition or division". That is an overgeneralization of distribution of logarithm symbol over addition or division. That also happens over any of subtraction and multiplication. These errors are examples of DOIF with logarithm function.

Pattern 5: trigonometric function symbol distributivity over addition and subtraction

(Fischbein, 1994; Matz, 1980; Olivier, 1989) pointed to students' errors in trigonometric functions as: *sin* (a + b) = *sin* a + *sin* b

(Mansoor, 2002) showed that senior primary and even secondary schools; students have difficulties in trigonometric functions, and do errors as:

1. *sin* (x+y) = *sin* x + *sin* y,
2. *cos* (x-y) = *cos* x - *cos* y.

3. $sin\ (75) = sin\ (45) + sin\ (30) = \frac{1}{\sqrt{2}} + \frac{1}{2}$

The above wrong statements can be assimilating; Table 5 includes analysis of the statements; *sin* (x+y) = *sin* x + *sin* y, *cos* (x-y) = *cos* x - *cos* y.

Table 5. Analysis of students' substituting errors in trigonometric functions

Error	No. 1	No. 2
dealing	*sin* (x+y) = *sin* x + *sin* y	*cos* (x-y) = *cos* x - *cos* (y
similar	a(b+c) = (ab)+(ac)	a(b - c) = ab – ac
Verbal description	Distributive of function symbol over addition	Distributive of function symbol over subtraction.
Symbolic description	f(a+b) = f(a) + f(b)	f(a-b) = f(a) - f(b)
analysis	Distributivity overgeneralization	

Table 5 shows that, the two wrong statements: *sin* (x+y) = *sin* x + *sin* y, *cos* (x-y) = *cos* x - *cos* y are including "trigonometric function symbol distributivity over any of addition and subtraction". That is an overgeneralization of distribution of trigonometric symbol over any of addition and subtraction. These errors are examples of DOIF with trigonometric function symbol.

Pattern 6: polynomial function symbol distributivity over four basic operations

(Mansoor, 2002) showed that senior primary and secondary schools' students always suffer from difficulties in Substituting with values to Polynomial functions, and do errors as:

1. $f\ (x) = x^2 + 15\,x + 7$, Then, f (3a÷4) =?

$f(3a \div 4) = \{ (3a)^2 + 15(3a) + 7\} \div \{ (4)^2 + 15(4) + 7\}$ =

$= \{ 9\,a^2 + 45a\} \div \{14 + 60 + 7\}$

$= \{ 9\,a^2 + 45a\} \div \{81\} = \frac{1}{9}\{ a^2 + 5a\}$

2. $f(x) = 14x^2 + 5x + 7$, Then $f(2x) \times f(3)$ =?
$f(2x) \times f(3) = f(6x) = 14(6x)^2 + 5(6x) + 7 = 504x^2 + 30x + 7$
3. $f(x) = 8x^2 + x + 3$, Then, $f(5a+10) = f(5x) + f(10)$ =?
$f(5x + 10) = \{8(5a)^2 + (5a) + 3\} + \{8(10)^2 + (10) + 3\}$
$= \{200a^2 + 5a + 3\} + 800 + 10 + 3\} = 200a^2 + 5a + 816$

these errors may be become obvious by the next display of the incorrect solution to the first question.
$f(x) = x^2 + 15x + 7$, Then, $f(3a \div 4)$ = ?

The first procedure was: $f(3a \div 4) = \{(3a)^2 + 15(3a) + 7\} \div \{(4)^2 + 15(4) + 7\}$

Table 6 contains analogy of this procedure assimilating it to multiplication Distributivity over addition and subtraction in numbers.

Table 6. Analogy of substituting errors in polynomial functions.

	The beginning	**Procedure's details**							**abstract**
Student procedure	**f (3a÷ 4)**	**$= \{(3a)^2 + 15(3a) + 7\} \div \{(4)^2 + 15(4) + 7\}$**						**=**	**f (3a) ÷ f(4)**
Coding of Student procedure	**f (a÷b)**	**= f**	**(**	**a**	**÷**	**b**	**)**	**=**	**f (a) ÷ f (b)**
Classic similar form	**a (b+c)**	**= a**	**(**	**b**	**+**	**c**	**)**	**=**	**a (b) + a (c)**

This analogy of students procedure means that $f(3x \div 4) = f(3x) \div f(4)$; this is similar to the form a (b+c). this means that function symbol is distributive left- over of division. That is inappropriate.

All of the above wrong statements have similar procedures, regardless of the difference between the substitute values.

These errors mean that some students distribute "polynomial function" symbol over addition, subtraction, multiplication and division of substitutive values.

These errors indicate distributivity overgeneralization when substituting values with Polynomial function. This is a pattern of DOIF in substituting values with Polynomial function.

Pattern 7: division distributivity by rational functions

Senior primary and secondary school students have difficulties in dividing any of constant, linear, quadric, cubic or any other rational function , and do errors (Mansoor, 2002) as:

1. $\frac{3}{(2x-3)} = \frac{3}{2x} - 1$
2. $\frac{6x^4}{(2x^4+3x^2+6)} = 3 + 2x^2 + x^4$

These treatments of the division can be assimilated.

Table 7 includes analysis of these divisions: $\frac{3}{(2x-3)} = \frac{3}{2x} - 1$, $\frac{6x^4}{(2x^4+3x^2+6)} = 3 + 2x^2 + x^4$

Table 7. Analysis of substituting errors in division distributivity by Polynomial functions.

Error	No. 1	No. 2
Dealing	$\frac{3}{(2x-3)} = \frac{3}{2x} - 1$	$\frac{6x^4}{(2x^4+3x^2+6)} = 3 + 2x^2 + x^4$
Similar	a÷(b-c) = (a÷b)-(a÷c)	a÷ (b+c+d) = (a÷b)+(a÷c)+ (a÷d)
Verbal description	Division distributive left- over subtraction of divisors.	Division distributive left- over addition of divisors.
Analysis	Distributivity overgeneralization	

Table 7 shows that, the two wrong statements above are including "dividend distributive left over addition and subtraction of divisors”. These errors are examples of DOIF in dealing with Division by Polynomial functions.

Pattern 8: absolute value function symbol distributivity over addition and subtraction:

(Mansoor, 2002) showed that secondary schools’ students have difficulties in absolute value functions, and do errors as:

1. |x-y| = |x| - |y| ; where x and y are real numbers, then x>0, y<0
2. |2354.543 + (245.152) + (-2.4523)| = |2354.543 | + |245.152| + |-2354.543|

The above two statements are assimilated in table 8.

Table 8. Analysis of adding or subtraction in dealing with absolute value function

Dealing	|x-y| = |x| - |y|	|2354.543 + (245.152) + (-2.4523)| = |2354.543 | + |245.152| + |-2354.543|
Similar	a(b-c) = (ab)-(ac)	a(b + c + d) = ab + ac +ad
Verbal description	Distributive of Absolute value function symbol over subtraction.	Distributive of Absolute value function symbol over addition
Symbolic description	f(a-b) = f(a) - f(b)	f(a+b+c) = f(a) + f(b) + f(c)
Analysis	Distributivity overgeneralization	

Table 8 shows that, the two wrong statements: |x-y| = |x| - |y|,

|2354.543 + (245.152) + (-2.4523)| = |2354.543 | + |245.152| + |-2354.543| are including “Absolute value function symbol distributivity over any of addition and subtraction”. That is an overgeneralization of distribution of Absolute value symbol over any of addition and subtraction. These errors are examples of DOIF in absolute value function.

Discussion

Literature review showed eight Patterns of distribution overgeneralization inside functions. Table 9 displays error pattern in each function.

Table 9. patterns of distribution overgeneralization inside functions

Function	Pattern
Power function	Distributive of the exponent over addition of the base
Exponential function	Distributive of the base over division of the exponent
Root function	Radical sign distributivity over each of addition and subtraction of radicands in root function
Logarithm function	Distributive of logarithm symbol over addition.
Trigonometric function	Trigonometric function symbol distributivity over addition and subtraction
Polynomial function	Polynomial function symbol distributivity over four basic operations
Rational function	Division left- over distributivity by rational functions
Absolute value function	Absolute value function symbol distributivity over addition and subtraction

Why these overgeneralizations of distributivity happen?

These error patterns happen consciously for two reasons:

1- All of eight overgeneralizations patterns can be seen similar to the distributive property upon numbers. These patterns in functions are similar to one of the two forms of multiplication Distributivity:

$m(a+b) = ma + mb$ or, $(a+b)m = am + bm$

literature review (Egodawatte & Stoilescu, 2015; Fischbein, 1994; Fischbein & Barash, 1993; Kirshner, 1985; Laursen, 1978; Mansoor, 2002; Matz, 1980; Olivier, 1989; Schwartzman, 1977) shows how far these error patterns are overgeneralizations of distributive property.

2- Frequent exposing (familiarity) to these experiences of distributive in different subjects of mathematics. previously the researcher pointed to various situations in primary, senior primary and secondary schools.

The similarity of distributive property in these various subjects with these situations that dealing with functions and the frequent exposing to situations about realizing distributive in many subjects and many operations are enough for Impact transmission of the learning.

These situations about division binary operations in secondary school indicate that learner apply distributive in so many binary operations, and this applying distributive property becomes more familiar as the learner moves from grade to upper and from educational stage to the next.

Previously the researcher discussed some of literature about overgeneralizations & Impact transmission of the learning. this discussion indicated that the impact transmission increases with the similarities between two or more situations. thus overgeneralizations are a product of an impact transmission of the learning. the probability of overgeneralizations increases with the similarities between various situations.

In short, familiarity with the distributive properties in more binary operations and more mathematical subjects lead to distributive overgeneralizations inside functions.

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